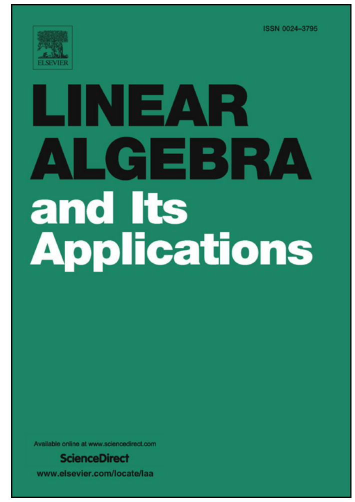

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A NOTE ON THE CONVEXITY OF THE MOORE–PENROSE INVERSE

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ABSTRACT. This note is a sequel to an earlier study (Nordström [7]) on convexity properties of the inverse and Moore–Penrose inverse, in which the following question was raised. Given nonnegative definite matrices A and B with Moore–Penrose inverses A^+ and B^+, respectively, can one show that

$$(\lambda A + \overline{\lambda} B)^+ \leqslant \lambda A^+ + \overline{\lambda} B^+$$

holding for a single $\lambda \in]0, 1[$ is enough to guarantee its validity for all $\lambda \in]0, 1[$? [The ordering above is the partial ordering, induced by the convex cone of nonnegative definite matrices, and $\overline{\lambda} := 1-\lambda$.] In this note an affirmative answer is provided to this question.

1. Introduction

Let $\mathbb{C}^{n\times m}$ denote the set of $n \times m$ matrices over the complex field $\mathbb{C}$, and let $\mathcal{H}^n$, $\mathcal{H}^n_{\geqslant}$, and $\mathcal{H}^n_{>}$ denote, respectively, the set of Hermitian, nonnegative definite, and positive definite matrices of order $n \times n$. Given $C \in \mathbb{C}^{n\times m}$, let C^*, C^+, and $\mathcal{R}(C)$ stand for the conjugate transpose, Moore–Penrose inverse, and range, respectively, of C. For $H, K \in \mathcal{H}^n$, write $H \leqslant K$ (or $K \geqslant H$) if $K - H \in \mathcal{H}^n_{\geqslant}$; this is the natural (Loewner) partial ordering of $\mathcal{H}^n$, induced by the closed convex cone $\mathcal{H}^n_{\geqslant}$.

It is well known that, on the set of positive definite matrices, the matrix inverse is a convex map, i.e., for every $A, B \in \mathcal{H}^n_{>}$ and for every $\lambda \in]0, 1[$, we have

$$(\lambda A + \overline{\lambda} B)^{-1} \leqslant \lambda A^{-1} + \overline{\lambda} B^{-1}, \tag{1.1}$$

with $\overline{\lambda} := 1 - \lambda$. (For the history of this matrix/operator inequality, see Nordström [7] and the references therein.)

One approach to showing inequalities of the type (1.1) is to first show it for the case $\lambda = 1/2$, and to then use continuity to conclude that it

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