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Extended commutation principles for normal decomposition systems

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ABSTRACT

In this paper we study some optimization problems on Lewis' normal decomposition systems (1996, 2000). We extend Gowda and Jeong's commutation principle (2017) in two directions. Firstly, we replace the sum operation by an arbitrary binary operation monotone in each variable. Secondly, we employ G-increasing functions in place of G-invariant convex functions. We give applications to some classes of matrices. We also interpret the obtained results for simple Euclidean Jordan algebras.

1. Introduction and summary

In this paper we consider optimization problems for some classes of composition functions defined on a normal decomposition system (NDS) [9,10]. We provide a necessary condition for a point in a NDS to be a minimizer/maximizer of such a function.

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Iusem and Seeger [8] proved the so-called *commutation principle* (CP) for the space of real symmetric matrices of size $n \times n$. Ramírez, Seeger and Sossa [16] extended CP to Euclidean Jordan algebras. In [6] Gowda and Jeong showed CP in the context of general normal decomposition systems.

Our motivation is to develop a unified framework for analyzing some optimization problems in a wide range of situations. We extend Gowda and Jeong's commutation principle (2017) in two directions (see Section 3). Firstly, we replace the sum operation by an arbitrary binary operation L monotone in each variable. Secondly, we employ G-increasing functions in place of G-invariant convex functions. Some examples for matrix subspaces are provided in Section 4. We also give interpretations of the obtained results for simple Euclidean Jordan algebras (see Section 5).

The structure of the paper is as follows. In Section 2 we present some basics of group majorization preorderings and group induced cone preorderings. Normal decomposition systems, due to Lewis (1996, 2000), are also demonstrated. Some matrix examples of NDS are given, too.

Roughly speaking, a NDS (V, G, γ) is an inner product space V endowed with the action of a compact group G of orthogonal operators on the space and with the eigenvalue-type function $\gamma(\cdot)$ such that some special generalizations of the Spectral Theorem and of Fan–Theobald's inequality are satisfied.

In context of future applications, we also introduce simultaneously diagonalizable pairs of vectors and the notion of commutativity of pairs of vectors in a NDS (see [6, p. 1397]).

We begin Section 3 with the definition and examples of two variables functions L monotone in each variable. We focus on the problem of minimizing the composition function $x \to L(\Theta(x), F(x))$, where x runs over a G-invariant and convex subset of V with (V, G, γ) being a NDS. We assume that F is G-increasing and Θ is differentiable in Gateaux sense. It is our aim to give a necessary condition for a point in V to be a minimizer for this function. In Theorem 3.5 we show that such minimizer must commute with the minus gradient of Θ at this point. We also present a variant of such theorem for the plus gradient of Θ (see Remark 3.6). Next we discuss analogous problem of maximizing the composition function (see Remarks 3.7–3.8).

In the rest of this section we give interpretations of Theorem 3.5 and Remarks 3.6–3.8 for L representing four binary arithmetic actions: $+$, $-$, $\cdot$ and $/$ (see Corollaries 3.12, 3.16–3.18 for details). We provide Corollaries 3.9, 3.10 and 3.13 for Θ being a linear or quadratic form. Here we recover some results from [6] and [8]. In Examples 3.11, 3.14 and 3.15 we investigate (i) the problem of operator norm of a linear map, (ii) cone complementary problem, and (iii) the problem of the maximal eigenvalue of a positive definite symmetric matrix.

In Section 4 we illustrate the obtained results in matrix context. Here we employ some NDSs related to Hermitian matrices, complex matrices and real skew-symmetric matrices, respectively. In Example 4.2 we demonstrate the problem of finding an Hermitian matrix with maximal distance from some G-invariant convex set. A similar problem for arbitrary complex matrices is considered in Example 4.5. In Example 4.3, we present the