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Polarity and separation of cones

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Given a closed convex cone $C \subset \mathbb{R}^n$ and its polar cone C°, properties of the set $C \cap (-C^\circ)$ are studied. In particular, we solve a problem of Stoker concerning nonemptiness of $\operatorname{rint} C \cap (-\operatorname{rint} C^\circ)$. Based on these properties, new results on separation of C and C° by hyperplanes are established.

1. Introduction

We recall that a nonempty set C in the n-dimensional Euclidean space $\mathbb{R}^n$ is a *cone* if $\lambda x \in C$ whenever $\lambda \geqslant 0$ and $x \in C$. (Obviously, this definition implies that the origin o of $\mathbb{R}^n$ belongs to C, although a stronger condition $\lambda > 0$ can be beneficial; see, e.g., [6].) The cone C is called *convex* if it is a convex set. In a standard way, the (negative) *polar cone* C° of C is defined by

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$$C^\circ = \{e \in \mathbb{R}^n : x{\cdot}e \leqslant 0 \ \text{ for all } \ x \in C\},$$

where $x{\cdot}e$ means the dot (scalar) product of vectors x and e.

Despite a wide usage of polar cones C and C° in various mathematical disciplines, not much is known about the sets

$$D = C \cap (-C^\circ) \quad \text{and} \quad E = \operatorname{rint} C \cap (-\operatorname{rint} C^\circ),$$

where $\operatorname{rint} C$ denotes the relative interior of C (we observe the set $-C^\circ$ is often called the dual cone of C).

Blumenthal [1] and Dines [2] showed that the existence of positive solutions of certain systems of homogeneous linear inequalities can be geometrically formulated as the property $D \neq \{o\}$ of a suitable convex cone $C \subset \mathbb{R}^n$. In this regard Gaddum [4], using a simple argument, proved that $D \neq \{o\}$ if and only if the cone C is not a subspace.

Independently, Stoker [9] asked whether the set E is nonempty and gave a partial affirmative answer by proving that $\operatorname{rint} C \cap (-C^\circ) \neq \varnothing$ for the case when C is a closed convex cone in $\mathbb{R}^3$, distinct from a subspace. Another motivation for the study of the sets D and E comes from separation theory. For instance, an assertion of Klee [5] on the existence of a hyperplane specially separating closed convex cones C_1 and C_2 in $\mathbb{R}^n$ satisfying the condition $C_1 \cap C_2 = \{o\}$ can be equivalently reformulated as $\operatorname{rint} C_1^\circ \cap (-\operatorname{rint} C_2^\circ) \neq \varnothing$ (see Theorem 4.2 below).

In this paper, we determine the dimensions of the sets D and E and prove that $\operatorname{rint} D = E$, as shown in Theorems 3.1 and 3.2. These results allow us to answer affirmatively Stoker's question on the nonemptiness of E (see Corollary 3.1).

The concluding Section 4 contains new assertions on separation of arbitrary convex cones C_1 and C_2 in $\mathbb{R}^n$ (which refine a result of Klee [5]), and, in particular, on separation of cones C and C° (see Theorems 4.1–4.3 and their corollaries)

2. Notation, terminology, and preliminaries

We follow standard notation and terminology of finite-dimensional convex analysis (see, for instance, the books [7] and [8]). In particular, $\operatorname{cl} F$, $\dim F$, $\operatorname{rbd} F$, $\operatorname{rint} F$, and $\operatorname{span} F$ stand, respectively, for the closure, dimension, relative boundary, relative interior, and span of a convex set $F \subset \mathbb{R}^n$.

For a simplicity of language, we will be dealing with *closed* convex cones in $\mathbb{R}^n$. Indeed, the obtained results can be easily rewritten for the case of any convex cones, based on the equalities $\operatorname{rint} C = \operatorname{rint}(\operatorname{cl} C)$ and $C^\circ = (\operatorname{cl} C)^\circ$.

Given a closed convex cone $C \subset \mathbb{R}^n$, the set $\operatorname{lin} C = C \cap (-C)$ is called the *lineality space* of C, and C is called *pointed* provided $\operatorname{lin} C = \{o\}$. It is known that $\operatorname{lin} C$ is the largest subspace contained in C and $C = C + \operatorname{lin} C$ (see, e.g., [8], Theorems 4.14 and 4.15). Obviously, $C \neq \operatorname{lin} C$ if and only if C is not a subspace. Equivalently, the number

$$s(C) = \dim C - \dim(\operatorname{lin} C) \tag{1}$$