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Computing the cardinality of the lattice of characteristic subspaces

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We obtain the cardinality of the lattice of characteristic subspaces of a nilpotent Jordan matrix when the underlying field is $GF(2)$, the only field where the lattices of characteristic and hyperinvariant subspaces can be different. If the characteristic polynomial of the matrix splits in the field, the general case can be reduced to the nilpotent Jordan case. Results are complex and highly combinatorial, and include the design of an algorithm.

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1. Introduction

Let $\mathbb{F}$ be a field and $A \in M_n(\mathbb{F})$ a matrix. A vector subspace $V \subseteq \mathbb{F}^n$ is called invariant with respect to A if $AV \subseteq V$. The subspace V is hyperinvariant if it is invariant for every matrix commuting with A, and it is characteristic if it is invariant for every nonsingular matrix commuting with A.

We denote by $\text{Hinv}(A)$ and $\text{Chinv}(A)$ the lattices of hyperinvariant and characteristic subspaces of A, respectively. Properties of these lattices have been analyzed (see [8,3,4], and the more recent papers [1,2,6,7]).

In this paper we determine the cardinality of the lattice of characteristic subspaces of a nilpotent Jordan matrix J. Assuming that the characteristic polynomial of J splits in the underlying field, this covers the most general case. If $\mathbb{F} \neq GF(2)$, then $\text{Chinv}(J) = \text{Hinv}(J)$ (see [1]), and the cardinality of $\text{Hinv}(J)$ is already known and easy to compute (see [3]). If $\mathbb{F} = GF(2)$, a condition by Shoda ([8]) characterizes when $\text{Hinv}(J)$ and $\text{Chinv}(J)$ do not coincide.

As $\text{Hinv}(J) \subseteq \text{Chinv}(J)$, we understand

$$\text{Chinv}(J) = \text{Hinv}(J) \cup (\text{Chinv}(J) \setminus \text{Hinv}(J)).$$

Our aim is to obtain the cardinality of $\text{Chinv}(J) \setminus \text{Hinv}(J)$ working on $\mathbb{F} = GF(2)$. For that purpose, we use the characterization of the subspaces in $\text{Chinv}(J) \setminus \text{Hinv}(J)$ obtained in [6]. According to [6], a characteristic nonhyperinvariant subspace X of J can be written as a direct sum of two subspaces $X = Y \oplus Z$, where Z and Y are associated to a so called char-tuple, Y is hyperinvariant with some extra conditions and Z is called a minext subspace. In our approach, we find the number of possible char-tuples, and the number of minext and hyperinvariant subspaces associated to each char-tuple. We obtain the number of minext subspaces through a recurrent formula, and the number of hyperinvariant subspaces associated to a char-tuple, through an algorithm.

The results obtained are much more complex than in the hyperinvariant case. They involve combinatorial numbers and Gauss binomial coefficients. The algorithm constructs a table which generalizes the Pascal matrix.

The paper is structured as follows: In Section 2, we recall basic definitions and introduce notation and previous results. In Section 3, we recall the characterization of characteristic nonhyperinvariant subspaces obtained in [6]. The cardinality of $\text{Chinv}(J) \setminus \text{Hinv}(J)$ and the above mentioned algorithm are obtained in Section 4. Finally, in Section 5 it is shown that the results of the algorithm can also be derived from generating polynomials.

2. Preliminaries

Let $\mathbb{F}$ be a field. Let $\mathbb{F}^n$ be the n-dimensional vector space over $\mathbb{F}$ and $A \in M_n(\mathbb{F})$ a square matrix corresponding to an endomorphism of $\mathbb{F}^n$ in a fixed basis.