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Multivariate Riordan groups and their representations [☆]

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Riordan bases in a power series ring of several variables are introduced as a special type of Schauder bases. As a part of the method of generating differentials, interplay of Riordan bases generalizes Lagrange inversion formulas. The set of Riordan bases has a group law with a specified set of variables as the unit element. Riordan arrays appear in a representation of the group.

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1. Introduction

Riordan Groups were first introduced by L.W. Shapiro, S. Getu, W.-J. Woan and L.C. Woodson [12]. They proved that certain infinite lower triangular matrices, called Riordan arrays, given by two power series in one variable form a group under the matrix multiplication. Along with many applications, the structure of these matrices has been analyzed. For instance, A-sequences and Z-sequences are used to characterize Riordan arrays [5]. There are also various generalizations of Riordan arrays in one or several variables [1,4,13,14]. Extending several known results, this paper provides an intrinsic treatment to Riordan arrays in several variables. To build up the Riordan group, we start with the Lagrange group whose elements are (sets of) variables of a power series ring. The Riordan group is obtained from the Lagrange group by joining one more invertible power series to variables. Indeed, the Riordan group is the semidirect product of a subgroup and the Lagrange group.

In the literature, applications of Riordan arrays require manipulation of matrices. In our intrinsic approach, computations are carried by residue calculus, which are performed in terms of variables of power series rings. The invariance of residues for chosen variables encompasses Lagrange inversion formulas [6]. For the case of one variable, a Schauder basis represents power series in a unique way. It is observed that inverse relations are just interplay of Schauder bases [8]. Riordan arrays come from a special type of Schauder bases, in which residue calculus is available. We believe this is the reason that Riordan arrays have many applications.

The definition of Schauder bases can be carried over from one variable to several variables without extra efforts. In this paper, a Riordan array in one or more variables is considered as a realization of certain Schauder basis through a group representation. More precisely, a Riordan basis, denote by $(G, \mathbf{X})$, is a Schauder basis given by an invertible power series G and a set of variables $\mathbf{X}$. With a fixed Riordan basis $(1, \mathbf{Z})$ serving as the unit element, we can define a product $(G, \mathbf{X}) * (H, \mathbf{Y})$ for Riordan bases so that they form a group. This group is called the Riordan group with respect to $\mathbf{Z}$. Riordan arrays appear as infinite matrices in a representation of this group.

Examining the ideas behind Schauder bases and Riordan arrays, we see differences between the method of generating functions and the method enhanced by differentials. For the method of generating differentials, the reader is referred to [9]. Here we compare the two methods to clarify our viewpoint and for future developments.

- *Non-canonical vs. Fixed Choice.* A power series with coefficients in a field κ can be characterized as a commutative Noetherian local ring, which is complete, regular and containing κ as a coefficient field. Generalizing variables, a Schauder basis can be used to represent power series. The method of generating differentials stands on the fact that there is no canonical choice of variables neither a Schauder basis. However, a set of variables are fixed for the theory of Riordan arrays or the method of generating functions.