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Topological foundations of an asymptotic approximation theory for sequences of matrices with increasing size

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Dedicated to Albrecht Böttcher, Carla Manni and Stefano Serra-Capizzano, the full professors who have supported me since the beginning of my research activity

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ABSTRACT

Sequences of matrices with increasing size arise in several contexts, including the discretization of integral and differential equations. An asymptotic approximation theory for this kind of sequences has recently been developed, with the aim of providing tools for computing their asymptotic singular value and eigenvalue distribution. The basis of this theory is the notion of approximating classes of sequences (a.c.s.), which is also fundamental to the theory of generalized locally Toeplitz sequences and hence to the spectral analysis of PDE discretization matrices. In this paper we show that the a.c.s. notion is a convergence notion induced by a pseudometrizable topology. We also identify a corresponding pseudometric and we study some of its properties. It turns out that there exists a strong connection between the a.c.s. topology and the topology of convergence in measure.

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1. Introduction

Let $\mathbb{C}^{n\times n}$ be the space of complex $n \times n$ matrices. Throughout this paper, a matrix-sequence is a sequence of the form $\{A_n\}_n$ with $A_n \in \mathbb{C}^{n\times n}$. Matrix-sequences naturally arise in several contexts. For example, when discretizing a linear differential or integral equation by a linear numerical method (such as the finite difference method, the finite element method, the modern isogeometric analysis, etc.), the actual computation of the numerical solution reduces to solving a linear system $A_n\mathbf{u}_n = \mathbf{g}_n$. The size n of this system diverges to infinity as the mesh discretization parameter tends to 0, and we are then in the presence of a matrix-sequence $\{A_n\}_n$. It is often observed in practice that $\{A_n\}_n$ belongs to the class of the so-called Generalized Locally Toeplitz (GLT) sequences, and in particular it enjoys an asymptotic singular value and eigenvalue distribution as $n \to \infty$; see [8,9,11,18,19,21] for more on this subject. Another noteworthy example concerns the finite sections of an infinite Toeplitz matrix. An infinite Toeplitz matrix is a matrix of the form

$$[a_{i-j}]_{i,j=1}^{\infty} = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & \cdots & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & \cdots & \cdots \\ a_2 & a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ \vdots & a_2 & a_1 & \ddots & \ddots & \ddots \\ \vdots & \vdots & a_2 & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{1}$$

The nth section of (1) is the $n \times n$ matrix defined by

$$A_n = [a_{i-j}]_{i,j=1}^{n}.$$

In the case where the entries a_k are the Fourier coefficients of a function $a \in L^1([-\pi,\pi])$, the matrix A_n is denoted by $T_n(a)$ and is referred to as the nth Toeplitz matrix generated by a. The asymptotic singular value and eigenvalue distribution of the matrix-sequence $\{T_n(a)\}_n$ has been deeply investigated in recent times, starting from Szegő's first limit theorem [5,14,25] and the Avram–Parter theorem [1,5,16], up to the works by Tyrtyshnikov–Zamarashkin [23,24,26] and Tilli [20,22].

In the last 20 years, an asymptotic approximation theory for matrix-sequences has been developed. The aim was to obtain sufficiently powerful tools for computing the asymptotic singular value and eigenvalue distribution of a 'difficult' matrix-sequence $\{A_n\}_n$ from the asymptotic singular value and eigenvalue distributions of 'simpler' matrix-sequences $\{B_{n,m}\}_n$ that 'converge' to $\{A_n\}_n$ in a suitable way as $m \to \infty$. The cornerstone of all this approximation theory is the notion of approximating classes of sequences, which we report in Definition 1. This notion is due to Serra-Capizzano [17], but the underlying idea was already present in Tilli's pioneering paper on Locally Toeplitz