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Matrix semigroups with constant spectral radius ☆

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ABSTRACT

Multiplicative matrix semigroups with constant spectral radius (c.s.r.) are studied and applied to several problems of algebra, combinatorics, functional equations, and dynamical systems. We show that all such semigroups are characterized by means of irreducible ones. Each irreducible c.s.r. semigroup defines walks on Euclidean sphere, all its nonsingular elements are similar (in the same basis) to orthogonal. We classify all nonnegative c.s.r. semigroups and arbitrary low-dimensional semigroups. For higher dimensions, we describe five classes and leave an open problem on completeness of that list. The problem of algorithmic recognition of c.s.r. property is proved to be polynomially solvable for irreducible semigroups and undecidable for reducible ones.

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1. Introduction

Matrix semigroups with constant spectral radius, or semigroups of matrices bounded from above and from below, have many remarkable properties and naturally appear in problems of combinatorics, dynamical systems, functional equations, matrix algebras, etc. The variety of those semigroups is very rich and the problem of their classification is still open. In this paper, we solve this problem in low dimensions and for nonnegative semigroups of arbitrary dimensions. In general case, we show that any semigroup with constant spectral radius defines walks on an ellipsoid. This fact, which is of independent interest, gives a way to recognize such semigroups by a polynomial time algorithm. In the end of the paper, we apply our results to several problems from applications.

For the sake of simplicity, in this paper we focus on matrices over $\mathbb{R}$. Thus, we consider multiplicative closed semigroups of real $d \times d$ matrices. For a nonempty compact family $\mathcal{A}$ of $d \times d$ matrices, we denote by $\mathcal{S}_{\mathcal{A}}$ the semigroup generated by $\mathcal{A}$ by multiplications and taking closure.

Definition 1. A matrix semigroup $\mathcal{S}$ has constant spectral radius (in short, $\mathcal{S}$ is c.s.r.) if the spectral radius of all its elements is the same and nonzero. A compact matrix family $\mathcal{A}$ is called c.s.r. if it generates a c.s.r. semigroup $\mathcal{S}_{\mathcal{A}}$.

The spectral radius $\rho(A)$ of a matrix A is the maximal modulus of its eigenvalues. Clearly, in any c.s.r. semigroup the spectral radius of all matrices is equal to one. Indeed, if for some $A \in \mathcal{S}$, we have $\rho(A) = c$, then $\rho(A^2) = c^2$. On the other hand, the semigroup $\mathcal{S}$ is c.s.r. and $A^2 \in \mathcal{S}$, hence $c^2 = c$, and so $c = 1$.

Definition 2. A family of matrices $\mathcal{A}$ is irreducible if there is no proper linear subspace of $\mathbb{R}^d$ invariant for all matrices from $\mathcal{A}$.

The semigroup $\mathcal{S}_{\mathcal{A}}$ generated by a family $\mathcal{A}$ is irreducible precisely when so is $\mathcal{A}$. Semigroups with constant spectral radius have several important characteristic properties listed below. Some of them concern only irreducible semigroups.

1. If $\mathcal{S}$ is a semigroup with multiplicative spectral radius, i.e., $\rho(AB) = \rho(A)\rho(B)$, $A, B \in \mathcal{S}$, then the semigroup $\{[\rho(A)]^{-1}A \mid A \in \mathcal{S},\ \rho(A) \neq 0\}$ has constant spectral radius. Thus, the study of semigroups with multiplicative spectral radius is essentially reduced to c.s.r. semigroups. This observation was put to good use in [24]. Moreover, for irreducible semigroups, the submultiplicativity of the spectral radius ($\rho(AB) \leq \rho(A)\rho(B)$, $A, B \in \mathcal{S}$) is equivalent to its multiplicativity [16, Theorem 2.1]. This property of semigroups can be relaxed further to the so-called Rota condition [24, Theorem 4.3].
2. All finite matrix semigroups that do not contain zero matrices are c.s.r. For integer matrices, the converse is also true: every irreducible c.s.r. semigroup of integer matrices is finite. We analyse this aspect in more detail in Subsection 9.1.