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# On the dual formulation of obstacle problems for the total variation and the area functional

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**Abstract**

We investigate the Dirichlet minimization problem for the total variation and the area functional with a one-sided obstacle. Relying on techniques of convex analysis, we identify certain dual maximization problems for bounded divergence-measure fields, and we establish duality formulas and pointwise relations between (generalized) BV minimizers and dual maximizers. As a particular case, these considerations yield a full characterization of BV minimizers in terms of Euler equations with a measure datum. Notably, our results apply to very general obstacles such as BV obstacles, thin obstacles, and boundary obstacles, and they include information on exceptional sets and up to the boundary. As a side benefit, in some cases we also obtain assertions on the limit behavior of $p$-Laplace type obstacle problems for $p \searrow 1$.

On the technical side, the statements and proofs of our results crucially depend on new versions of Anzellotti type pairings which involve general divergence-measure fields and specific representatives of BV functions. In addition, in the proofs we employ several fine results on (BV) capacities and one-sided approximation.
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*Keywords:* (Thin) obstacle problem; Total variation; Convex duality; Optimality conditions; Anzellotti pairing; BV capacity

## 1. Introduction

**Obstacle problems for total variation and area functional.** In this paper, on a bounded open set $\Omega \subset \mathbb{R}^n$ of dimension $n \in \mathbb{N}$, we study certain variational problems with unilateral obstacles. More precisely, our primary interest is in the minimization problem for the total variation

$$\int_\Omega |\mathrm{D}u|\,\mathrm{d}x \tag{1.1}$$

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among functions $u\colon \Omega \to \mathbb{R}$ which satisfy, for a given obstacle $\psi$, the zero Dirichlet boundary condition and the obstacle constraint

$$u = 0 \qquad \text{on } \partial\Omega\,, \tag{1.2}$$

$$u \geq \psi \qquad \text{on } \Omega\,. \tag{1.3}$$

We directly remark that instead of (1.2) we will eventually consider the non-homogeneous condition $u = u_o$ on $\partial\Omega$ with a quite general Dirichlet datum $u_o$, but for the purposes of this introductory exposition we limit ourselves to the homogeneous case in (1.2). Moreover, imposing the same constraints on $u$, we also study the minimization problem for the area integral

$$\int_\Omega \sqrt{1+|\mathrm{D}u|^2}\,\mathrm{d}x\,, \tag{1.4}$$

which, for sufficiently smooth functions $u$, gives the $n$-dimensional area of the graph of $u$.

It is well known that these problems are naturally set in the space $\mathrm{BV}(\Omega)$ of functions of bounded variation and that, thanks to weak$*$ compactness, general existence results for BV minimizers can be obtained. Indeed, such results involve the BV versions $|\mathrm{D}u|(\overline{\Omega})$ and $\sqrt{1+|\mathrm{D}u|^2}(\overline{\Omega})$ of the functionals in (1.1) and (1.4), where in order to explain[1] $\mathrm{D}u$ also on $\partial\Omega$ one extends $u$ by 0 outside $\Omega$. Moreover, one imposes mild assumptions on the obstacle $\psi$ to ensure compatibility of (1.2) and (1.3) and make sure that the admissible class is non-empty ($\psi \in \mathrm{L}^\infty_{\mathrm{cpt}}(\Omega)$ suffices, but can be weakened). Then, if the obstacle condition (1.3) is understood as an $\mathcal{L}^n$-a.e. inequality, the existence of a minimizer follows by a standard application of the direct method. In particular, the validity of (1.3) is preserved in this reasoning, since BV embeds compactly in $\mathrm{L}^1$ and minimizing sequences possess $\mathcal{L}^n$-a.e. convergent subsequences.

We record that, under specific assumptions on $\psi$, some finer existence and regularity results for the problems in (1.1)–(1.4) have been obtained in [24–26,37], for instance. However, in the present paper, we are concerned with different issues, which are approached in a general setting with much lighter assumptions on $\psi$.

**Thin obstacles and relaxation.** With regard to the case of thin obstacles (i.e. obstacles which are positive only on $(n-1)$-dimensional surfaces), it is also interesting to understand (1.3) as an a.e. condition with respect to the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$. This understanding of the constraint is also natural, since $\mathcal{H}^{n-1}$-a.e. means quasi everywhere with respect to 1-capacity; cf. (2.15). The resulting point of view leads to a more general theory and typically makes an essential difference for obstacles which are — as it occurs in the thin case — upper semicontinuous, but neither continuous nor BV. Precisely, as a substitute for (1.3) we thus employ the condition

$$u^+ \geq \Psi \qquad \mathcal{H}^{n-1}\text{-a.e. on } \Omega\,, \tag{1.5}$$

where the capital letter $\Psi$ is used for the $\mathcal{H}^{n-1}$-a.e. defined obstacle and the approximate upper limit $u^+$ gives the largest one among the reasonable $\mathcal{H}^{n-1}$-a.e. defined representatives of $u \in \mathrm{BV}(\Omega)$; see Section 2.3 for the precise definition. Under (1.5) the existence issue becomes more subtle, since a (minimizing) sequence of smooth functions $u_k \geq \Psi$ may converge to a limit $u \in \mathrm{BV}(\Omega)$ which does not anymore satisfy (1.5) — even though the usage of $u^+$ means that (1.5) is understood in the widest possible $\mathcal{H}^{n-1}$-a.e. sense. This difficulty can be overcome by passing to suitable relaxations (or more precisely to $\mathrm{L}^1$-lower semicontinuous envelopes) of the functionals (1.1) and (1.4), and indeed explicit formulas for the relaxations — when starting from competitors $w \in \mathrm{W}^{1,1}_0(\Omega)$ with (1.5) — have been identified by Carriero & Dal Maso & Leaci & Pascali [15]. Their corresponding result [15, Theorem 7.1] states that, if $\Omega$ has a Lipschitz boundary and a Borel function $\Psi$ on $\Omega$ is compatible with (1.2) in the sense that there exists a competitor $w \in \mathrm{W}^{1,1}_0(\Omega)$ with (1.5), then the relaxed functionals on $\mathrm{BV}(\Omega)$ are given by

$$\mathrm{TV}_{\Psi;\Omega}(u) = |\mathrm{D}u|(\overline{\Omega}) + \int_\Omega (\Psi - u^+)_+\,\mathrm{d}\varsigma\,, \tag{1.6}$$

[1] To be precise, the above quantities are defined for all $u \in \mathrm{BV}(\Omega)$ such that the extension with value 0 on $\mathbb{R}^n \setminus \Omega$ is BV on the whole $\mathbb{R}^n$, and the measure $\mathrm{D}u$ then represents the distributional gradient of the extended function. Correspondingly, the quantities $|\mathrm{D}u|(\overline{\Omega})$ and $\sqrt{1+|\mathrm{D}u|^2}(\overline{\Omega})$ are understood as the total variations over $\overline{\Omega}$ of this $\mathbb{R}^n$-valued measure $\mathrm{D}u$ and the $\mathbb{R}^{1+n}$-valued measure $(\mathcal{L}^n, \mathrm{D}u)$ (with the Lebesgue measure $\mathcal{L}^n$), and they suitably take into account the zero Dirichlet condition in (1.2).

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