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Divergence-free positive symmetric tensors and fluid dynamics

Denis Serre

École Normale Supérieure de Lyon, U.M.P.A., UMR CNRS–ENSL # 5669, 46 allée d'Italie, 69364 Lyon cedex 07, France

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Abstract

We consider $d \times d$ tensors $A(x)$ that are symmetric, positive semi-definite, and whose row-divergence vanishes identically. We establish sharp inequalities for the integral of $(\det A)^{\frac{1}{d-1}}$. We apply them to models of compressible inviscid fluids: Euler equations, Euler–Fourier, relativistic Euler, Boltzman, BGK, etc. We deduce an *a priori* estimate for a new quantity, namely the space–time integral of $\rho^{\frac{1}{n}} p$, where ρ is the mass density, p the pressure and n the space dimension. For kinetic models, the corresponding quantity generalizes Bony's functional.

Keywords: Conservation laws; Gas dynamics; Functional inequalities

Notations. The integer $d \geq 2$ is the number of independent variables, which are often space–time coordinates. It serves also for the size of square matrices. If $1 \leq j \leq d$ and $x \in \mathbb{R}^d$ are given, we set $\widehat{x_j} = (\dots, x_{j-1}, x_{j+1}, \dots) \in \mathbb{R}^{d-1}$; the projection $x \mapsto \widehat{x_j}$ ignores the j-th coordinate. The transpose of a matrix M is M^T. If $A \in \mathbf{M}_d(\mathbb{R})$, its cofactor matrix $\widehat{A}$ satisfies

$$\widehat{A}^T A = A\,\widehat{A}^T = (\det A) I_d, \qquad \det \widehat{A} = (\det A)^{d-1}.$$

Because we shall deal only with symmetric matrices, we have simply $\widehat{A} A = A \widehat{A} = (\det A) I_d$. The space of $d \times d$ symmetric matrices with real entries is $\mathbf{Sym}_d$. The cones of positive definite, respectively positive semi-definite, matrices are $\mathbf{SPD}_d$ and $\mathbf{Sym}_d^+$. If $u \in \mathbb{R}^d$, $u \otimes u \in \mathbf{Sym}_d^+$ denotes the rank-one matrix of entries $u_i u_j$.

The unit sphere of $\mathbb{R}^d$ is S^{d-1}. The Euclidean volume of an open subset Ω of $\mathbb{R}^d$ is denoted $|\Omega|$. If the boundary $\partial\Omega$ is rectifiable, we denote the same way $|\partial\Omega|$ its area, and $ds(x)$ the area element. For instance, the ball B_r of radius r and its boundary, the sphere S_r, satisfy $|B_r| = \frac{r}{d}|S_r|$. If Ω has a Lipschitz boundary, its outer unit normal $\vec{n}$ is defined almost everywhere.

If $f : \Omega \to \mathbb{R}$ is integrable, its average over Ω is the number

$$\fint_\Omega f(x)\,dx := \frac{1}{|\Omega|} \int_\Omega f(x)\,dx.$$

E-mail address: denis.serre@ens-lyon.fr.

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Given a lattice Γ of $\mathbb{R}^d$, and $f : \mathbb{R}^d \to \mathbb{R}$ a Γ-periodic, locally integrable function, we denote

$$\int_{\mathbb{R}^d/\Gamma} f(x)\,dx$$

the value of the integral of f over any fundamental domain. We define as above the average value

$$⨍_{\mathbb{R}^d/\Gamma} f(x)\,dx.$$

For our purpose, a *tensor* is a matrix-valued function $x \mapsto T(x) \in \mathbf{M}_{p\times q}(\mathbb{R})$. If $q = d$ and if the derivatives make sense (say as distributions), we form

$$\mathrm{Div}T = \left(\sum_{j=1}^{d} \partial_j t_{ij}\right)_{1\le i\le p},$$

which is vector-valued. We emphasize the uppercase letter D in this context. We reserve the lower case operator **div** for vector fields.

If $1 \le p \le \infty$, its conjugate exponent is p'.

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1. Introduction

We first define the mathematical object under consideration.

Definition 1.1. Let Ω be an open subset of $\mathbb{R}^d$. A *divergence-free positive symmetric tensor* (in short, a *DPT*) is a locally integrable tensor $x \mapsto A(x)$ over Ω with the properties that $A(x) \in \mathbf{Sym}_d^+$ almost everywhere, and $\mathrm{Div}\, A = 0$.

The following fact is obvious.

Lemma 1.1 *(Congruence.). If A is a DPT and $P \in \mathbf{GL}_d(\mathbb{R})$ is given, then the tensor*

$$B(y) := PA(P^{-1}y)P^T$$

is also a DPT.

1.1. Motivations: Where do the divergence-free positive symmetric tensors occur?

Most of our examples, though not all of them, come from fluid dynamics, where a DPT contains a stress tensor.

Compressible gas. In space dimension $n \ge 1$, a gas is described by a mass density $\rho \ge 0$, a velocity u and a pressure $p \ge 0$. These fields obey the Euler equations (conservation of mass and momentum)

$$\partial_t\rho + \mathrm{div}_y(\rho u) = 0, \qquad \partial_t(\rho u) + \mathrm{Div}_y(\rho u \otimes u) + \nabla_y p = 0.$$

Here $x = (t, y)$ and $d = 1 + n$. The tensor

$$A(t, y) = \begin{pmatrix} \rho & \rho u^T \\ \rho u & \rho u \otimes u + pI_n \end{pmatrix}$$

is a DPT.

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