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Well-posedness for mean-field evolutions arising in superconductivity

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with an appendix jointly written with Julian Fischer [c]

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Abstract

We establish the existence of a global solution for a new family of fluid-like equations, which are obtained in certain regimes in [24] as the mean-field evolution of the supercurrent density in a (2D section of a) type-II superconductor with pinning and with imposed electric current. We also consider general vortex-sheet initial data, and investigate the uniqueness and regularity properties of the solution. For some choice of parameters, the equation under investigation coincides with the so-called lake equation from 2D shallow water fluid dynamics, and our analysis then leads to a new existence result for rough initial data.

1. Introduction

1.1. General overview

We study the well-posedness of the following two fluid-like evolution models coming from the mean-field limit equations of Ginzburg–Landau vortices: first, for $\alpha \geq 0$, $\beta \in \mathbb{R}$, we consider the "incompressible" flow

$$\partial_t v = \nabla P - \alpha(\Psi + v)\operatorname{curl} v + \beta(\Psi + v)^\perp \operatorname{curl} v, \qquad \operatorname{div}(av) = 0, \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^2, \tag{1.1}$$

and second, for $0 \leq \lambda < \infty$, $\alpha > 0$, $\beta \in \mathbb{R}$, we consider the "compressible" flow

$$\partial_t v = \lambda \nabla(a^{-1}\operatorname{div}(av)) - \alpha(\Psi + v)\operatorname{curl} v + \beta(\Psi + v)^\perp \operatorname{curl} v, \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^2, \tag{1.2}$$

with $v : \mathbb{R}^+ \times \mathbb{R}^2 := [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}^2$ and $\operatorname{curl} v \geq 0$, where $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ is a given forcing vector field, and where the weight $a := e^h$ is determined by a given "pinning potential" $h : \mathbb{R}^2 \to \mathbb{R}$. Note that the incompressible model (1.1)

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can be seen as the limiting case $\lambda = \infty$ of the family of compressible models (1.2). As established in a companion paper [24] with Sylvia Serfaty, these equations are obtained in certain regimes as the mean-field evolution of the supercurrent density in a (2D section of a) type-II superconductor described by the 2D Ginzburg–Landau equation with pinning and with imposed electric current — but without gauge and in whole space, for simplicity. In this context, the cases $\lambda = \infty$, $0 < \lambda < \infty$, and $\lambda = 0$ correspond respectively to a low, an intermediate, and a high vortex density regime. Note that in the parabolic case $\alpha > 0$, $\beta = 0$, the incompressible model (1.1) can be seen as a Wasserstein gradient flow for the vorticity $\operatorname{curl} v$, but a common gradient flow structure seems to be missing for the whole family of equations (1.2) with $\lambda \in [0, \infty]$. In the conservative case $\alpha = 0$ with $\Psi = 0$, the incompressible model (1.1) takes the form of the so-called lake equation from 2D shallow water fluid dynamics [26, p. 235] (see also [12,13]), which reduces to the usual 2D Euler equation if the weight a is constant.

In the nondegenerate case $\lambda > 0$, we investigate existence, uniqueness, and regularity, both locally and globally in time, for the Cauchy problems associated with (1.1) and (1.2), and we also consider vortex-sheet initial data. In Appendix A jointly written with Julian Fischer, a complete theory is further obtained for the degenerate parabolic case $\lambda = \beta = 0$, $\alpha > 0$.

1.2. Brief discussion of the model

Superconductors are materials that in certain circumstances lose their resistivity, which allows permanent supercurrents to circulate without loss of energy. In the case of type-II superconductors, if the external magnetic field is not too strong, it is expelled from the material (Meissner effect), while, if it is much too strong, the material returns to a normal state. Between these two critical values of the external field, these materials are in a mixed state, allowing a partial penetration of the external field through "vortices", which are accurately described by the (mesoscopic) Ginzburg–Landau theory. Restricting ourselves to a 2D section of a superconducting material, it is standard to study for simplicity the 2D Ginzburg–Landau equation on the whole plane (to avoid boundary issues) and without gauge (although the gauge is expected to bring only minor difficulties). We refer e.g. to [53,52] for further reference on these models, and to [47] for a mathematical introduction. In this framework, in the asymptotic regime of a large Ginzburg–Landau parameter (which is indeed typically the case in real-life superconductors), vortices are known to become point-like, and to interact with one another according to a Coulomb pair potential. In the mean-field limit of a large number of vortices, the evolution of the suitably normalized (macroscopic) mean-field density $\omega : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ of the vortex liquid was then naturally conjectured to satisfy the following Chapman–Rubinstein–Schatzman–E equation [25,16]

$$\partial_t \omega = \operatorname{div}(|\omega| \nabla (-\triangle)^{-1} \omega), \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^2, \tag{1.3}$$

where $(-\triangle)^{-1}\omega$ is indeed the Coulomb potential generated by the vortices. Although the vortex density ω is a priori a signed measure, we restrict here (and throughout this paper) to positive measures, $|\omega| = \omega \geq 0$, so that the above is replaced by

$$\partial_t \omega = \operatorname{div}(\omega \nabla (-\triangle)^{-1} \omega), \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^2. \tag{1.4}$$

More precisely, the mean-field supercurrent density $v : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2$ (linked to the vortex density through the relation $\omega = \operatorname{curl} v$) was conjectured to satisfy

$$\partial_t v = \nabla P - v \operatorname{curl} v, \qquad \operatorname{div} v = 0, \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^2. \tag{1.5}$$

Taking the curl of this equation indeed formally yields (1.4), noting that the incompressibility constraint $\operatorname{div} v = 0$ allows to write $v = \nabla^\perp \triangle^{-1} \omega$.

In the context of superfluidity [1,46], a conservative counterpart of the usual parabolic Ginzburg–Landau equation is used as a mesoscopic model. This counterpart is given by the Gross–Pitaevskii equation, which is a particular instance of a nonlinear Schrödinger equation. At the level of the mean-field evolution of the corresponding vortices, we then need to replace (1.3)–(1.4) by their conservative versions, thus replacing $\nabla(-\triangle)^{-1}\omega$ by $\nabla^\perp(-\triangle)^{-1}\omega$. As argued in [5], there is also physical interest in rather starting from the "mixed-flow" (or "complex") Ginzburg–Landau equation, which is a mix between the usual Ginzburg–Landau equation describing superconductivity ($\alpha = 1$, $\beta = 0$, below), and its conservative counterpart given by the Gross–Pitaevskii equation ($\alpha = 0$, $\beta = 1$, below). The above mean-field equation (1.5) for the supercurrent density v is then replaced by the following, for $\alpha \geq 0$, $\beta \in \mathbb{R}$,