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Uniform convergence for the incompressible limit of a tumor growth model

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Abstract

We study a model introduced by Perthame and Vauchelet [19] that describes the growth of a tumor governed by Brinkman's Law, which takes into account friction between the tumor cells. We adopt the viscosity solution approach to establish an optimal uniform convergence result of the tumor density as well as the pressure in the incompressible limit. The system lacks standard maximum principle, and thus modification of the usual approach is necessary.

Keywords: Viscosity solutions; Tumor growth; Front propagation

1. Introduction

We study the following model, which was introduced by Perthame and Vauchelet in [19]. It describes the growth of tumors at the cellular level by providing a law relating the cell density, pressure, and cell multiplication. The tumor cell density $n_k : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ satisfies,

$$\begin{cases} \partial_t n_k - \operatorname{div}(n_k DW_k) = n_k G(p_k), \\ -\nu \Delta W_k + W_k = p_k, \end{cases} \tag{1.1}$$

where the pressure p_k is given by,

$$p_k = \frac{k}{k-1}(n_k)^{k-1}.$$

Here ν is a positive constant and G is a given function that describes the effect that the pressure has on the growth of the tumor. We assume G satisfies,

$$G \in C^1(\mathbb{R}),\ G'(\cdot) \le -\bar{\alpha} < 0,\ \text{and } G(P_M) = 0 \text{ for some } P_M > 0 \text{ and } \bar{\alpha} > 0. \tag{1.2}$$

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The main results of [19] concern the limit as $k \to \infty$, or the so-called incompressible limit, of (1.1). This connects (1.1) to a system that involves a moving front. If the parameter ν were zero (in other words, if the tumor were governed by Darcy's Law), then the system (1.1) would become,

$$\partial_t n_k - \mathrm{div}(n_k Dp_k) = n_k G(p_k).$$

This model for tumor growth has been widely studied, and we refer the reader to the introduction of [19] for a variety of references, both about modeling and rigorous mathematical analysis. In particular, in [18], Perthame, Quiròs and Vázquez find that the incompressible limit of the above equation is the Hele–Shaw problem with a forcing term. Kim and Pozar [16] used viscosity solution methods to improve the result in [18]. The model that we study, (1.1) with $\nu > 0$, has been proposed as a better description of tumor growth. Here, the tumor is governed by Brinkman's Law, which takes into account the friction between the tumor cells, and not just of the tumor with its environment. These modeling issues are discussed in, for example, [24,6]. For fixed k, the system (1.1) was also studied by Trivisa and Webber in [23], who established existence of weak solutions and found a convergent numerical method for (1.1).

Of particular interest in the asymptotic limit is the limiting pressure, which represents the incompressibility condition. In the inviscid model ($\nu = 0$), the limiting pressure solves a Hele–Shaw type problem and is continuous as long as the pressure zone is reasonably regular [18]. However, as illustrated in [19], in the viscous model that we study here the limiting pressure is strictly positive on the boundary of its support, and thus is discontinuous. This is an interesting contrast to the inviscid model.

Our goal in this paper is to obtain pointwise convergence results in the framework of viscosity solutions theory, improving the L^1 convergence obtained in [19]. Due to the discontinuity of the limiting pressure, the optimal pointwise convergence result one expects is uniform convergence away from the pressure boundary. This is precisely what we obtain. In addition, knowing that the pressure converges uniformly then allows us to improve the convergence of the W_k as well (see Theorem 1.1 below).

We point out that the system (1.1) does not enjoy the comparison principle – in fact, it is strongly coupled – and thus one needs to modify the existing theory in the analysis. To achieve this we follow the approach in [15], where we rely on the fact that one component of the system can be considered almost fixed due to its strong convergence: in our case that turns out to be the W_k, though their convergence is still weaker than what is available in [15].

Heuristics. Let us briefly recall the formal derivation of the limiting system given in [19] to illustrate additional challenges and main ingredients of our analysis in more detail. We denote the limit of (p_k, n_k, W_k) by $(p_\infty, n_\infty, W_\infty)$. Perhaps the easiest equation to guess is the one for W_∞:

$$-\nu \Delta W_\infty + W_\infty = p_\infty. \tag{1.3}$$

Next we expect that p_∞ is either zero or satisfies $p_\infty - \nu G(p_\infty) = W_\infty$. This is because we can write the n_k equation in terms of p_k as

$$\partial_t p_k - Dp_k \cdot DW_k = (k-1)\nu^{-1} p_k (W_k - (Id - \nu G)(p_k)), \tag{1.4}$$

which then translates p_∞ as a singular limit of reaction–diffusion equations. Thus it is reasonable to think that p_∞ will take value either zero or $(Id - \nu G)^{-1}(W_\infty)$. In other words, we expect to have $p_\infty = (Id - \nu G)^{-1}(W_\infty)\chi_{\Omega_t}$ for some region Ω_t. The question now is to characterize Ω_t.

We recall that there is a third component here, namely n_k. Manipulating the equation for n_k and then using the equation that W_k satisfies yields,

$$\partial_t n_k - Dn_k \cdot DW_k = n_k(\Delta W_k + G(p_k)) = \frac{n_k}{\nu}(W_k - p_k + \nu G(p_k)). \tag{1.5}$$

The region Ω_t is where the p_k converge to the positive value $(Id - \nu G)^{-1}(W_\infty)$, so by definition we know that the n_k converge to 1 there. When the p_k converge to 0 (in other words, on Ω_t^c) we expect the n_k to converge to zero if initially this is the case (see the discussion in the outline below). Notice that in both situations, the right-hand side of the previous equation is zero. Thus we expect n_∞ to equal χ_{Ω_t} and solve,

$$\partial_t n_\infty - Dn_\infty \cdot DW_\infty = 0, \tag{1.6}$$

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