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Gagliardo–Nirenberg inequalities and non-inequalities: The full story

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Abstract

We investigate the validity of the Gagliardo–Nirenberg type inequality

$$\|f\|_{W^{s,p}(\Omega)} \lesssim \|f\|^{\theta}_{W^{s_1,p_1}(\Omega)} \|f\|^{1-\theta}_{W^{s_2,p_2}(\Omega)}, \tag{1}$$

with $\Omega \subset \mathbb{R}^N$. Here, $0 \le s_1 \le s \le s_2$ are non negative numbers (not necessarily integers), $1 \le p_1, p, p_2 \le \infty$, and we assume the standard relations

$$s = \theta s_1 + (1-\theta)s_2,\ 1/p = \theta/p_1 + (1-\theta)/p_2 \text{ for some } \theta \in (0,1).$$

By the seminal contributions of E. Gagliardo and L. Nirenberg, (1) holds when s_1, s_2, s are integers. It turns out that (1) holds for "most" of values of $s_1, \dots, p_2$, but not for all of them. We present an explicit condition on s_1, s_2, p_1, p_2 which allows to decide whether (1) holds or fails.

Keywords: Sobolev spaces; Gagliardo–Nirenberg inequalities; Interpolation inequalities

1. Introduction

In two seminal independent contributions, E. Gagliardo [8] and L. Nirenberg [10] established the interpolation inequality[1]

$$\|f\|_{W^{k,p}} \lesssim \|f\|^{\theta}_{W^{k_1,p_1}} \|f\|^{1-\theta}_{W^{k_2,p_2}},\ \forall f \in W^{k_1,p_1}(\mathbb{R}^N) \cap W^{k_2,p_2}(\mathbb{R}^N), \tag{1.1}$$

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[1] In Eq. (1.1), $A \lesssim B$ means $A \le CB$ for some positive constant C.

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where k_1, k_2, k are non negative integers and $1 \leq p_1, p_2, p \leq \infty$. These quantities are related by the standard relations

$$k = \theta k_1 + (1-\theta)k_2, \ \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2} \text{ and } 0 < \theta < 1. \tag{1.2}$$

We investigate the validity of the analogous inequality when the smoothness exponents k_1, k_2, k are not necessarily integers. More specifically, assume that the real numbers $0 \leq s_1, s_2, s$, $\theta \in (0, 1)$ and $1 \leq p_1, p_2, p \leq \infty$ satisfy the relations

$$s = \theta s_1 + (1-\theta)s_2, \ \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2} \text{ and } 0 < \theta < 1. \tag{1.3}$$

We ask whether the estimate

$$\|f\|_{W^{s,p}(\Omega)} \lesssim \|f\|^{\theta}_{W^{s_1,p_1}(\Omega)} \|f\|^{1-\theta}_{W^{s_2,p_2}(\Omega)}, \ \forall f \in W^{s_1,p_1}(\Omega) \cap W^{s_2,p_2}(\Omega) \tag{1.4}$$

holds. Here, Ω is a *standard domain* in $\mathbb{R}^N$, i.e.,

$$\Omega \text{ is either } \mathbb{R}^N \text{ or a half space or a Lipschitz bounded domain in } \mathbb{R}^N, \tag{1.5}$$

and $\|f\|_{W^{s,p}}$ denotes the usual Sobolev norm (see Section 2).

Let us note that (1.4) holds when $s_1 = s_2$; this is simply Hölder's inequality. In our analysis, we may thus assume that

$$s_1 < s < s_2. \tag{1.6}$$

It has been part of the folklore of the Sobolev spaces theory that (1.4) holds in "most" cases but fails in some "limiting" cases. For example if $0 < s_1 < s_2 < 1$, (1.4) is an immediate consequence of Hölder's inequality. While if $\Omega = (0, 1)$, $s_1 = 0$, $s_2 = 1$, $p_1 = \infty$, $p_2 = 1$, $\theta = 1/2$, (1.4) becomes

$$\|f\|_{H^{1/2}((0,1))} \lesssim \|f\|^{1/2}_{W^{1,1}((0,1))} \|f\|^{1/2}_{L^\infty((0,1))}, \ \forall f \in W^{1,1}((0,1)), \tag{1.7}$$

which implies

$$\|f\|_{H^{1/2}((0,1))} \lesssim \|f\|^{1/2}_{BV((0,1))} \|f\|^{1/2}_{L^\infty((0,1))}, \ \forall f \in BV((0,1)). \tag{1.8}$$

But (1.8) is clearly wrong (take e.g. $f = \mathbb{1}_{(0,1/2)}$), so that (1.7) also fails.

To the best of our knowledge, the precise "dividing line" between the "good" and the "bad" cases in (1.4) was never clarified. It is our goal to fill this gap.

The following condition plays an essential role.[2]

$$s_2 \text{ is an integer } \geq 1, \ p_2 = 1 \text{ and } s_2 - s_1 \leq 1 - \frac{1}{p_1}. \tag{1.9}$$

Here is our main result.

Theorem 1. *Inequality (1.4) holds if and only if (1.9) fails.*

More precisely, we have

A) *If (1.9) fails then, for every $\theta \in (0, 1)$, there exists a constant C depending on s_1, s_2, p_1, p_2, θ and Ω such that*

$$\|f\|_{W^{s,p}(\Omega)} \leq C \|f\|^{\theta}_{W^{s_1,p_1}(\Omega)} \|f\|^{1-\theta}_{W^{s_2,p_2}(\Omega)}, \ \forall f \in W^{s_1,p_1}(\Omega) \cap W^{s_2,p_2}(\Omega). \tag{1.10}$$

B) *If (1.9) holds there exists some $f \in W^{s_1,p_1}(\Omega) \cap W^{s_2,p_2}(\Omega)$ such that $f \notin W^{s,p}(\Omega)$, $\forall \theta \in (0, 1)$.*

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[2] The latter condition can also be written in the more symmetric form $s_1 - \dfrac{1}{p_1} \geq s_2 - \dfrac{1}{p_2}$.