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# Martingale–coboundary decomposition for families of dynamical systems

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**Abstract**

We prove statistical limit laws for sequences of Birkhoff sums of the type $\sum_{j=0}^{n-1} v_n \circ T_n^j$ where $T_n$ is a family of nonuniformly hyperbolic transformations.

The key ingredient is a new martingale–coboundary decomposition for nonuniformly hyperbolic transformations which is useful already in the case when the family $T_n$ is replaced by a fixed transformation $T$, and which is particularly effective in the case when $T_n$ varies with $n$.

In addition to uniformly expanding/hyperbolic dynamical systems, our results include cases where the family $T_n$ consists of intermittent maps, unimodal maps (along the Collet–Eckmann parameters), Viana maps, and externally forced dispersing billiards.

As an application, we prove a homogenisation result for discrete fast–slow systems where the fast dynamics is generated by a family of nonuniformly hyperbolic transformations.


*Keywords:* Martingale–coboundary decomposition; Nonuniform hyperbolicity; Statistical limit laws; Homogenisation; Fast–slow systems

## 1. Introduction

The emergence of statistical and stochastic phenomena in deterministic dynamical systems is currently a very active area. Topics of sustained interest include central limit theorems, invariance principles (weak and almost sure convergence to Brownian motion), moment estimates, and homogenisation (whereby deterministic systems with multiple timescales converge to a stochastic differential equation).

One of the standard techniques for investigating such phenomena is the martingale–coboundary decomposition method of Gordin [26] which has seen extensive development in both the probability theory literature (for example [31, 35,41,51]) and in the dynamical systems literature (for example [39,54,55]).

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In this paper, we introduce a new version of the Gordin method and show that it has significant advantages over previous versions when applied to a wide range of questions in nonuniformly hyperbolic dynamics. Even in the case of a single nonuniformly hyperbolic transformation, there are advantages to the new approach which seems more elementary and more powerful than the existing ones in the literature. In addition, our method is well suited for studying sequences of Birkhoff sums of the form $S_n = \sum_{j=0}^{n-1} v_n \circ T_n^j$ where $T_n^{j+1} = T_n^j \circ T_n$ which arise naturally in averaging and homogenisation problems. Here, $T_n : \Lambda_n \to \Lambda_n$, $n \ge 0$ is a sequence of measure-preserving transformations defined on probability spaces $(\Lambda_n, \mu_n)$. It is assumed that the transformations $T_n$ are nonuniformly expanding/hyperbolic with uniform constants, but no restrictions are imposed on how $T_n$ varies with $n$.

In the case of a single nonuniformly hyperbolic map $T$, our method applies directly to $T$ bypassing any induced limit theorems for the associated induced uniformly hyperbolic map. Unlike other approaches [35,39,41,54], no approximation arguments are required for the central limit theorem (CLT) and weak invariance principle (WIP) when the inducing time is not $L^3$. For moment estimates, the method does not require special arguments when the inducing time is not $L^2$ (cf. [22,30]). In addition, we obtain a simple proof of an unexpected CLT for systems with nonsummable decay of correlations due to [28], whereas the previous proof relied on operator renewal theory and the Wiener lemma in noncommutative Banach algebras.

Still in the case of a single map $T$, our method is very well-adapted for obtaining a secondary martingale–coboundary decomposition for the square of the martingale in the decomposition mentioned above. This enables control on sums of squares as is often required in more sophisticated limit laws. To illustrate this, we consider an almost sure invariance principle with excellent error rates due to [21], and show that our method of applying their results leads to stronger conclusions in certain examples.

The main advantage of the approach, however, is that it allows explicit control on various constants associated with each transformation $T$, making the method especially useful for sums of the form $\sum_{j=0}^{n-1} v_n \circ T_n^j$. This in turn has applications to fast–slow systems of the type considered in [37]. Whereas [37] obtained rates of averaging, we prove results here on homogenisation.

The remainder of this paper is organised as follows. In Section 2, we establish the new martingale–coboundary decomposition for nonuniformly expanding maps and show how this implies moment estimates and the WIP. In Section 3, we obtain a secondary martingale–coboundary decomposition and apply this to the almost sure invariance principle. In Section 4, we derive limit laws for families of nonuniformly expanding maps. This is extended to families of nonuniformly hyperbolic transformations in Section 5. In Section 6, we state and prove an abstract theorem on homogenisation for discrete time fast–slow systems, generalising [27]. In Section 7, we verify the hypotheses in Section 6 when the fast dynamics is given by a family of nonuniformly hyperbolic transformations.

**Notation.** We write $\to_{\mu_n}$ to denote weak convergence with respect to a specific family of probability measures $\mu_n$ on the left-hand side. So $A_n \to_{\mu_n} A$ means that $A_n$ is a family of random variables on $(\Lambda_n, \mu_n)$ and $A_n \to_w A$.

For $J \in \mathbb{R}^{m\times n}$, we write $|J| = \big(\sum_{i=1}^m \sum_{j=1}^n J_{ij}^2\big)^{1/2}$.

We use "big O" and $\ll$ notation interchangeably, writing $a_n = O(b_n)$ or $a_n \ll b_n$ if there is a constant $C > 0$ such that $a_n \le C b_n$ for all $n \ge 1$. As usual, $a_n = o(b_n)$ means that $\lim_{n\to\infty} a_n/b_n = 0$.

Recall that $v : \Lambda \to \mathbb{R}$ is a Hölder observable on a metric space $(\Lambda, d)$ if $\|v\|_\eta = |v|_\infty + |v|_\eta < \infty$ where $|v|_\infty = \sup_\Lambda |v|$, $|v|_\eta = \sup_{x\ne y} \frac{|v(x)-v(y)|}{d(x,y)^\eta}$.

## 2. Martingale–coboundary decomposition for nonuniformly expanding maps

In this section, we prove our main theoretical result on martingale–coboundary decomposition for nonuniformly expanding maps. The notion of nonuniformly expanding map is recalled in Subsection 2.1. The martingale–coboundary decomposition is carried out in Subsection 2.2. Subsection 2.3 shows how certain limit laws follow from this decomposition.

### *2.1. Nonuniformly expanding maps*

Let $(\Lambda, d_\Lambda)$ be a bounded metric space with finite Borel measure $\rho$ and let $T : \Lambda \to \Lambda$ be a nonsingular transformation. Let $Y \subset \Lambda$ be a subset of positive measure, and let $\alpha$ be an at most countable measurable partition of $Y$ with

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