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Singular integrals and a problem on mixing flows ☆

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Abstract

We prove a result related to Bressan's mixing problem. We establish an inequality for the change of Bianchini semi-norms of characteristic functions under the flow generated by a divergence free time dependent vector field. The approach leads to a bilinear singular integral operator for which we prove bounds on Hardy spaces. We include additional observations about the approach and a discrete toy version of Bressan's problem.

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1. Introduction

1.1. Mixing flows

We consider subsets A of $\mathbb{T}^d \equiv \mathbb{R}^d/\mathbb{Z}^d$. For $0 < r < 1/4$, $x \in \mathbb{R}^d$ let $B_r(x)$ denote the ball of radius r centered at x, with respect to the usual geodesic distance on $\mathbb{T}^d$. A measurable set $E \subset \mathbb{T}^d$ is *mixed at scale r,* with mixing constant $\kappa \in (0, 1/2)$, if

$$\kappa \leq \frac{|E \cap B_r(x)|}{|B_r(x)|} \leq 1-\kappa, \quad \forall x \in \mathbb{T}^d. \tag{1}$$

Let v be a time-dependent, a priori smooth vector field, defined on $\mathbb{T}^d \times [0, T]$ with values in the tangent bundle of the torus. The vector field can be considered a vector field $(x, t) \mapsto v(x, t)$ on $\mathbb{R}^d$ which is periodic in x, i.e.

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$$v(x+k,t) = v(x,t) \text{ for all } (x,t) \in \mathbb{R}^d \times \mathbb{R},\ k \in \mathbb{Z}^d.$$

We assume that

$$\mathrm{div}_x v(x,t) = 0$$

and let Φ be the flow generated by v. I.e. Φ satisfies

$$\frac{\partial}{\partial t}\Phi(x,t) = v(\Phi(x,t),t),$$
$$\Phi(x,0) = x.$$

For every t the map $x \mapsto \Phi(x,t)$ is a volume preserving diffeomorphism on $\mathbb{R}^d$ satisfying

$$\Phi(x+k,t) - k = \Phi(x,t), \quad x \in \mathbb{R}^d, k \in \mathbb{Z}^d.$$

In what follows we shall also use the notation $\Phi_t(x) = \Phi(x,t)$. We are interested in mixing flows which transport an unmixed set Ω at time $t=0$ to a set $\Phi_T(\Omega)$ mixed at scale ε at time $t=T$.

1.2. Bressan's problem

Split $\mathbb{T}^d$ as $\Omega_L \cup \Omega_R$ with $\Omega_R = \Omega_L^\complement$ where

$$\Omega_L = \{x : 0 \le x_1 < \frac{1}{2}\}, \quad \Omega_R = \{x : \frac{1}{2} \le x_1 < 1\}. \tag{2}$$

Let $0 < \varepsilon < 1/4$. Consider a periodic flow Φ_t generated by a smooth time dependent divergence free vector field, and *assume that at time $t=T$ the flow mixes Ω_L at scale ε;* i.e. the set $E = \Phi_T(\Omega_L)$ satisfies (1) with $r = \varepsilon$. Bressan [5] asks (setting $\kappa = 1/3$) whether there is a universal constant $c_d > 0$ such that

$$\int_0^T \int_{[0,1)^d} |D_x v(x,t)|\,dx\,dt \ge c_d \log(1/\varepsilon). \tag{3}$$

As noted in [5] it suffices to consider the case $T=1$, by replacing $v(x,t)$ with $Tv(x,t/T)$. In [4], Bressan formulated a more general conjecture for mildly compressible flows.

Bressan's conjecture is still open at the time of this writing. Therefore it is of interest to ask for corresponding lower bounds if the $L^1(\mathbb{T}^d)$ norm is replaced by a larger norm. That is, under the assumption that the flow generated by v mixes the set at scale ε with mixing constant γ, do we have a universal lower bound of the form

$$\int_0^T \|D_x v(\cdot,t)\|_{\mathcal{Y}}\,dt \ge c_{\mathcal{Y}}(\kappa)\log(1/\varepsilon) \tag{4}$$

for suitable function spaces $\mathcal{Y} \subset L^1(\mathbb{T}^d)$ or even $\mathcal{Y} \subset M(\mathbb{T}^d)$ with $M(\mathbb{T}^d)$ the space of bounded Borel measures on $\mathbb{T}^d$? Crippa and De Lellis [7] showed this for $\mathcal{Y} = L^p(\mathbb{T}^d)$, $1 < p < \infty$ and also for the space $\mathcal{Y}$ consisting of functions for which the Hardy–Littlewood maximal function $M_{HL}f$ belongs to $L^1(\mathbb{T}^d)$, i.e. for $\mathcal{Y} = L\log L(\mathbb{T}^d)$. We shall discuss two ways to improve $\mathcal{Y}$ to a local Hardy space. In §7 we consider a discrete toy problem on $\mathbb{T}^2$ for which we prove an analogue of the L^1 conjecture, although this toy model does not yield significant information for the general Bressan problem. It should be noted that the lower bound $\log(1/\varepsilon)$ is sharp and cannot even be improved by working with L^p spaces, see the recent results by Yao and Zlatoš [21] and by Alberti, Crippa and Mazzucato [1].

1.3. An approach to Bressan's problem via a Bianchini semi-norm

We denote by

$$⨍_{B_r(x)} f(y)dy = \frac{1}{|B_r(x)|}\int_{B_r(x)} f(y)dy$$