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Existence of discretely self-similar solutions to the Navier–Stokes equations for initial value in $L^2_{loc}(\mathbb{R}^3)$

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Abstract

We prove the existence of a forward discretely self-similar solutions to the Navier–Stokes equations in $\mathbb{R}^3 \times (0, +\infty)$ for a discretely self-similar initial velocity belonging to $L^2_{loc}(\mathbb{R}^3)$.

MSC: 76B03; 35Q31

Keywords: Navier–Stokes equations; Existence; Discretely self-similar solutions

1. Introduction

In this paper we study the existence of forward discretely self-similar (DSS) solutions to the Navier–Stokes equations in $Q = \mathbb{R}^3 \times (0, +\infty)$

$$\nabla \cdot u = 0, \tag{1.1}$$

$$\partial_t u + (u \cdot \nabla)u - \Delta u = -\nabla \pi, \tag{1.2}$$

with the initial condition

$$u = u_0 \quad \text{on} \quad \mathbb{R}^3 \times \{0\}. \tag{1.3}$$

Here $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ denotes the velocity of the fluid, and $u_0(x) = (u_{0,1}(x), u_{0,2}(x), u_{0,3}(x))$, while π stands for the pressure. In case $u_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$ in the sense of distributions the global in time existence of weak solutions to (1.1)–(1.3), which satisfy the global energy inequality for almost all $t \in (0, +\infty)$

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2 ds \le \frac{1}{2}\|u_0\|_2^2 \tag{1.4}$$

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has been proved by Leray [9]. On the other hand, the important questions of regularity and uniqueness of solutions to (1.1)–(1.3) are still open. The first significant results in this direction have been established by Scheffer [10] and later by Caffarelli, Kohn, Nirenberg [2] for solutions (u, π) that also satisfy the following local energy inequality for almost all $t \in (0, +\infty)$ and for all nonnegative $\phi \in C^\infty_c(Q)$

$$\frac{1}{2}\int\limits_{\mathbb{R}^3} |u(t)|^2\phi(x,t)dx + \int\limits_0^t\int\limits_{\mathbb{R}^3} |\nabla u|^2\phi dxds$$
$$\le \frac{1}{2}\int\limits_0^t\int\limits_{\mathbb{R}^3} |u|^2\Big(\frac{\partial}{\partial t}+\Delta\Big)\phi dxds + \frac{1}{2}\int\limits_0^t\int\limits_{\mathbb{R}^3} (|u|^2+2\pi)u\cdot\nabla\phi dxds. \tag{1.5}$$

On the other hand, the space $L^2(\mathbb{R}^3)$ excludes homogeneous spaces of degree -1 belonging to the scaling invariant class. In fact we observe that $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$ solves the Navier–Stokes equations with initial velocity $u_{0,\lambda}(x) = \lambda u_0(\lambda x)$, for any $\lambda > 0$. This suggests to study of the Navier–Stokes system for initial velocities in a homogeneous space X of degree -1, which means that $\|v\|_X = \|v_\lambda\|_X$ for all $v \in X$. Koch and Tataru proved in [7] that $X = BMO^{-1}$ is the largest possible space with scaling invariant norm which guarantees well-posedness under smallness condition. On the contrary, for self-similar (SS) initial data fulfilling $u_{0,\lambda} = u$ for all $\lambda > 0$ a natural space seems to be $X = L^{3,\infty}(\mathbb{R}^3)$. This space is embedded into the space $L^2_{uloc}(\mathbb{R}^3)$, which contains uniformly local square integrable functions. Obviously, possible solutions to the Navier–Stokes equations with $u_0 \in L^2_{uloc}(\mathbb{R}^3)$ do not satisfy the global energy equality, rather the local energy inequality in the sense of Caffarelli–Kohn–Nirenberg. Such solutions are called local Leray solutions. The existence of global in time local Leray solutions has been proved by Lemariè-Rieusset in [8] (see also in [6] for more details). This concept has been used by Bradshaw and Tsai [1] for the construction of a discretely self-similar (λ-DSS, $\lambda > 1$) local Leray solution for a λ-DSS initial velocity $u_0 \in L^{3,\infty}(\mathbb{R}^3)$. This result generalizes the previous results of Jia and Šverák [5] concerning the existence of SS local Leray solution, and the result by Tsai in [11], which proves the existence of a λ-DSS Leray solution for λ near 1. However, for the λ-DSS initial data it would be more natural to assume $u_0 \in L^2_{loc}(\mathbb{R}^3)$ instead $L^{3,\infty}(\mathbb{R}^3)$. In general, such initial value does not belong to $L^2_{uloc}(\mathbb{R}^3)$ and therefore it does not belong to the Morrey class $M^{2,1}$, rather to the weighted space $L^2_k(\mathbb{R}^3)$ of all $v \in L^2_{loc}(\mathbb{R}^3)$ such that $\frac{v}{(1+|x|^k)} \in L^2(\mathbb{R}^3)$ for all $\frac{1}{2} < k < +\infty$.

Since the authors in [1] work on the existence of periodic solutions to the time dependent Leray equation a certain spatial decay is necessary which can be ensured for initial data in $L^{3,\infty}(\mathbb{R}^3)$. On the other hand, applying the local L^2 theory it would be more natural to assume $u_0 \in L^2(B_\lambda \setminus B_1)$ only. As explained in [1] their method even breaks down for initial data in the Morrey class $M^{2,1}(\mathbb{R}^3)$, which is a much smaller subspace of $L^2_{loc}(\mathbb{R}^3)$. By using an entirely different method we are able to construct a global weak solutions for such DSS initial data.

In the present paper we introduce a new notion of a local Leray solution satisfying a local energy inequality with projected pressure. To the end, we provide the notations of function spaces which will be used in the sequel. By $L^s(G)$, $1 \le s \le \infty$, we denote the usual Lebesgue spaces. The usual Sobolev spaces are denoted by $W^{k,s}(G)$ and $W^{k,s}_0(G)$, $1 \le s \le +\infty$, $k \in \mathbb{N}$. The dual of $W^{k,s}_0(G)$ will be denoted by $W^{-k,s'}(G)$, where $s' = \frac{s}{s-1}$, $1 < s < +\infty$. For a general space of vector fields X the subspace of solenoidal fields will be denoted by X_σ. In particular, the space of solenoidal smooth fields with compact support is denoted by $C^\infty_{c,\sigma}(\mathbb{R}^3)$. In addition we define the energy space

$$V^2(G\times(0,T)) = L^\infty(0,T;L^2(G)) \cap L^2(0,T;W^{1,2}(G)), \quad 0 < T \le +\infty.$$

We now recall the definition of the local pressure projection $E^*_G : W^{-1,s}(G) \to W^{-1,s}(G)$ for a given bounded C^2-domain $G \subset \mathbb{R}^3$, introduced in [13] based on the unique solvability of the steady Stokes system (cf. [4]). More precisely, for any $F \in W^{-1,s}(G)$ there exists a unique pair $(v,p) \in W^{1,s}_{0,\sigma}(G) \times L^s_0(G)$ which solves weakly the steady Stokes system

$$\begin{cases} \nabla\cdot v = 0 \quad \text{in} \quad G, \quad -\Delta v + \nabla p = F \quad \text{in} \quad G,\\ v = 0 \text{ on } \quad \partial G. \end{cases} \tag{1.6}$$

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