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ANIHPC:2857

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Constructing center-stable tori

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Received 26 October 2016; received in revised form 6 July 2017; accepted 31 July 2017

Abstract

We show that certain derived-from-Anosov diffeomorphisms on the 2-torus may be realized as the dynamics on a center-stable or center-unstable torus of a 3-dimensional strongly partially hyperbolic system. We also construct examples of center-stable and center-unstable tori in higher dimensions.

Keywords: Partial hyperbolicity; Dynamical coherence; Invariant submanifolds

1. Introduction

Partially hyperbolic dynamical systems have received a large amount of attention in recent years. These systems display a wide variety of highly chaotic behavior [2], but they have enough structure to allow, in some cases, for the dynamics to be understood and classified [5,13].

A diffeomorphism f is *strongly partially hyperbolic* if there is a splitting of the tangent bundle into three invariant subbundles $TM = E^u \oplus E^c \oplus E^s$ such that the derivative Df expands vectors in the unstable bundle E^u, contracts vectors in stable bundle E^s, and these dominate any expansion or contraction in the center direction E^c. (See section 2 for a precise definition.) The global properties of these systems are often determined by analyzing invariant foliations tangent to the subbundles of the splitting.

The bundles E^u and E^s are uniquely integrable [14]. That is, there are foliations W^u and W^s such that any curve tangent to E^u or to E^s lies in a single leaf of the respective foliation. For the center bundle E^c, however, the situation is more complicated. There may not be a foliation tangent to E^c. Even if such a foliation exists, the bundle may not be uniquely integrable since, in general, the center bundle is only Hölder continuous and not C^1 regular. The first discovered examples of partially hyperbolic systems without center foliations were algebraic in nature. In these examples, both f and the splitting can be taken as smooth, and the center bundle is not integrable because it does not satisfy Frobenius' condition of involutivity [18,20]. Such non-involutive examples are only possible if the dimension of the center bundle is at least two, and for a long time it was an open question if a one-dimensional center bundle was necessarily integrable.

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http://dx.doi.org/10.1016/j.anihpc.2017.07.005

Rodriguez Hertz, Rodriguez Hertz, and Ures recently answered this question by constructing a counterexample [17]. They defined a partially hyperbolic system on the 3-torus with a one-dimensional center bundle which does not integrate to a center foliation. In fact, the center bundle is uniquely integrable everywhere except for an invariant embedded 2-torus tangent to $E^c \oplus E^u$ where the center curves approaching from either side of the torus meet in cusps. This discovery has shifted our view on the possible dynamics a partially hyperbolic system can possess, and leads to questions of how commonly invariant submanifolds of this type occur in general. In this paper, we build further examples of partially hyperbolic systems having compact submanifolds tangent either to $E^c \oplus E^u$ or $E^c \oplus E^s$, both in dimension 3 and in higher dimensions.

In the construction in [17], the dynamics on the 2-torus tangent to $E^c \oplus E^u$ is Anosov. In fact, it is given by a hyperbolic linear map on $\mathbb{T}^2$, the cat map. It has long been known that a *weakly partially hyperbolic* system, that is, a diffeomorphism $g : \mathbb{T}^2 \to \mathbb{T}^2$ with a splitting of the form $E^c \oplus E^u$ or $E^c \oplus E^s$, need not be Anosov. Therefore, one can ask if a weakly partially hyperbolic system which is not Anosov may be realized as the dynamics on an invariant 2-torus sitting inside a 3-dimensional strongly partially hyperbolic system. We show, in fact, that derived-from-Anosov dynamics with sinks or sources may be realized on these tori.

To state the results, we say that diffeomorphisms f_0 and f_1 of the 2-torus are *dom-isotopic* if there an isotopy $\{f_t\}_{t\in[0,1]}$ such that every f_t has a dominated splitting.

Theorem 1.1. *Let $g : \mathbb{T}^2 \to \mathbb{T}^2$ be a weakly partially hyperbolic diffeomorphism which is dom-isotopic to a linear toral automorphism. Then, there is an embedding $i : \mathbb{T}^2 \to \mathbb{T}^3$ and a strongly partially hyperbolic diffeomorphism $f : \mathbb{T}^3 \to \mathbb{T}^3$ such that $i(\mathbb{T}^2)$ is a center-stable or center-unstable torus and $i^{-1} \circ f \circ i = g$.*

To be precise, a *center-stable torus* is an embedded copy of $\mathbb{T}^D$ with $D \geq 2$ which is tangent to $E^{cs}_f := E^c_f \oplus E^s_f$. Similarly, a *center-unstable torus* is tangent to $E^{cu}_f := E^c_f \oplus E^u_f$. We also use the terms cs-torus and cu-torus as shorthand.

If the diffeomorphism g has a weakly partially hyperbolic of splitting of the form $E^s_g \oplus E^c_g$, then $i(\mathbb{T}^2)$ will be a cs-torus. If the splitting is of the form $E^c_g \oplus E^u_g$, then $i(\mathbb{T}^2)$ will be a cu-torus. In the case where the derivative of g preserves the orientation of the center bundle, E^c_g, we may be more specific about the construction.

Theorem 1.2. *Let $g_0 : \mathbb{T}^2 \to \mathbb{T}^2$ be a weakly partially hyperbolic diffeomorphism which preserves the orientation of its center bundle and is dom-isotopic to a linear Anosov diffeomorphism $A : \mathbb{T}^2 \to \mathbb{T}^2$ and let $0 < \epsilon < \frac{1}{2}$. Then there is a strongly partially hyperbolic diffeomorphism $f : \mathbb{T}^3 \to \mathbb{T}^3$ such that*

(1) *$f(x,t) = (A(x), t)$ for all $(x,t) \in \mathbb{T}^3$ with $|t| > \epsilon$,*
(2) *$f(x,t) = (g_0(x), t)$ for all $(x,t) \in \mathbb{T}^3$ with $|t| < \frac{\epsilon}{2}$, and*
(3) *$\mathbb{T}^2 \times 0$ is either a center-stable or center-unstable torus.*

Since the construction is local in nature, different weakly partially hyperbolic diffeomorphisms may be inserted into the system at different places.

Corollary 1.3. *Suppose that $A : \mathbb{T}^2 \to \mathbb{T}^2$ is a hyperbolic linear automorphism and for each $i \in \{1, \dots, n\}$ that $g_i : \mathbb{T}^2 \to \mathbb{T}^2$ is a weakly partially hyperbolic diffeomorphism which preserves the orientation of its center bundle and is dom-isotopic to A. Let $\{t_1, \dots, t_n\}$ be a finite subset of the circle, S^1. Then there is a strongly partially hyperbolic diffeomorphism $f : \mathbb{T}^3 \to \mathbb{T}^3$ such that*

(1) *$f(x, t_i) = (g_i(x), t_i)$ for each t_i and all $x \in \mathbb{T}^2$, and*
(2) *each $\mathbb{T}^2 \times t_i$ is a center-stable or center-unstable torus.*

In the above theorems, the examples may be constructed in such a way that the resulting diffeomorphism $f : \mathbb{T}^3 \to \mathbb{T}^3$ is not dynamically coherent. See section 4 for details.

We also note that the presence of a cs or cu-torus affects the dynamics only in a neighborhood of that torus and does not place global restrictions on the dynamics on $\mathbb{T}^3$. For instance, one could easily construct a system which has a cs or cu-torus $\mathbb{T}^2 \times 0$ and has a robustly transitive blender elsewhere on $\mathbb{T}^3$ [1].

Please cite this article in press as: A. Hammerlindl, Constructing center-stable tori, Ann. I. H. Poincaré – AN (2017), http://dx.doi.org/10.1016/j.anihpc.2017.07.005