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ANIHPC:2860

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A classification result for the quasi-linear Liouville equation ☆

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Received 23 March 2017; received in revised form 20 July 2017; accepted 29 August 2017

Abstract

Entire solutions of the n-Laplace Liouville equation in $\mathbb{R}^n$ with finite mass are completely classified.

Keywords: Liouville equation; Quasilinear PDE; Classification; Entire solutions

1. Introduction

We are concerned with the following Liouville equation

$$\begin{cases} -\Delta_n U = e^U & \text{in } \mathbb{R}^n \\ \int_{\mathbb{R}^n} e^U < +\infty \end{cases} \tag{1.1}$$

involving the n-Laplace operator $\Delta_n(\cdot) = \mathrm{div}(|\nabla(\cdot)|^{n-2}\nabla(\cdot))$, $n \geq 2$. Here, a solution U of (1.1) stands for a function $U \in C^{1,\alpha}(\mathbb{R}^n)$ which satisfies

$$\int_{\mathbb{R}^n} |\nabla U|^{n-2}\langle \nabla U, \nabla \Phi\rangle = \int_{\mathbb{R}^n} e^U \Phi \qquad \forall\, \Phi \in H = \{\Phi \in W_0^{1,n}(\Omega) : \Omega \subset \mathbb{R}^n \text{ bounded}\}. \tag{1.2}$$

As we will see, the regularity assumption on U is not restrictive since a solution in $W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n)$ is automatically in $C^{1,\alpha}(\mathbb{R}^n)$, for some $\alpha \in (0,1)$.

Problem (1.1) has the explicit solution

$$U(x) = \log \frac{c_n}{(1+|x|^{\frac{n}{n-1}})^n}, \qquad x \in \mathbb{R}^n,$$

where $c_n = n(\frac{n^2}{n-1})^{n-1}$. Due to scaling and translation invariance, a $(n+1)$-dimensional family of explicit solutions $U_{\lambda,p}$ to (1.1) is built as

☆ Partially supported by Gruppo Nazionale per l'Analisi Matematica, la Probabilitá e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

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http://dx.doi.org/10.1016/j.anihpc.2017.08.002
0294-1449/

Please cite this article in press as: P. Esposito, A classification result for the quasi-linear Liouville equation, Ann. I. H. Poincaré – AN (2017), http://dx.doi.org/10.1016/j.anihpc.2017.08.002

$$U_{\lambda,p}(x) = U(\lambda(x-p)) + n\log\lambda = \log \frac{c_n \lambda^n}{(1+\lambda^{\frac{n}{n-1}}|x-p|^{\frac{n}{n-1}})^n} \tag{1.3}$$

for all $\lambda > 0$ and $p \in \mathbb{R}^n$. Notice that

$$\int_{\mathbb{R}^n} e^{U_{\lambda,p}} = \int_{\mathbb{R}^n} e^U = c_n \omega_n \tag{1.4}$$

where $\omega_n = |B_1(0)|$. Our aim is the following classification result:

Theorem 1.1. *Let U be a solution of (1.1). Then*

$$U(x) = \log \frac{c_n \lambda^n}{(1+\lambda^{\frac{n}{n-1}}|x-p|^{\frac{n}{n-1}})^n}, \quad x \in \mathbb{R}^n \tag{1.5}$$

for some $\lambda > 0$ and $p \in \mathbb{R}^n$.

In a radial setting Theorem 1.1 has been already proved, among other things, in [19]. For the semilinear case $n = 2$ such a classification result is known since a long ago. The first proof goes back to J. Liouville [29] who found a formula – the so-called Liouville formula – to represent a solution U on a simply-connected domain in terms of a suitable meromorphic function. On the whole $\mathbb{R}^2$ the finite-mass condition $\int_{\mathbb{R}^2} e^U < +\infty$ completely determines such meromorphic function.

A PDE proof has been found several years later by W. Chen and C. Li [9]. The fundamental point is to represent a solution U of (1.1) in an integral form in terms of the fundamental solution and then deduce the precise asymptotic behavior of U at infinity to start the moving plane technique. Such idea has revealed very powerful and has been also applied [7,27,30,40,41] to the higher-order version of (1.1) involving the operator $(-\Delta)^{\frac{n}{2}}$. Overall, the integral equation satisfied by U can be used to derive asymptotic properties of U at infinity or can be directly studied through the method of moving planes/spheres. Since these methods are very well suited for integral equations, a research line has flourished about qualitative properties of integral equations, see [10,18,24,42,43] to quote a few.

The quasi-linear case $n > 2$ is more difficult. Very recently, the classification of positive $\mathcal{D}^{1,n}(\mathbb{R}^N)$-solutions to $-\Delta_n U = U^{\frac{nN}{N-n}-1}$, a PDE with critical Sobolev polynomial nonlinearity, has been achieved [13,34,39] for $n < N$, see also some previous somehow related results [14,15,37]. The strategy is always based on the moving plane method and the analytical difficulty comes from the lack of comparison/maximum principles on thin strips. Moreover for $n < N$ it is not available any Kelvin type transform, a useful tool to "gain" decay properties on a solution.

When $n = N$ the classical approach [7,9,27,30,40,41] breaks down since an integral representation formula for a solution U of (1.1) is not available, due to the quasi-linear nature of Δ_n. It becomes a delicate issue to determine the asymptotic behavior of U at infinity and overall it is not clear how to carry out the method of moving planes/spheres. However, when $n = N$ there are some special features we aim to exploit to devise a new approach which does not make use of moving planes/spheres, providing in two dimensions an alternative proof of the result in [9]. During the completion of this work, we have discovered that such an approach has been already used in [8] for Liouville systems, where the maximum principle can possibly fail. See also [20,28] for a related approach to symmetry questions in a ball.

The case $n = N$ is usually referred to as the conformal situation, since Δ_n is invariant under Kelvin transform: $\hat{U}(x) = U(\frac{x}{|x|^2})$ formally satisfies

$$\Delta_n \hat{U} = \frac{1}{|x|^{2n}} (\Delta_n U)(\frac{x}{|x|^2}),$$

so that

$$\begin{cases} -\Delta_n \hat{U} = F(x) := \frac{e^{\hat{U}}}{|x|^{2n}} & \text{in } \mathbb{R}^n \setminus \{0\} \\ \int_{\mathbb{R}^n} \frac{e^{\hat{U}}}{|x|^{2n}} < +\infty. \end{cases}$$

The equation has to be interpreted in the weak sense