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On the asymptotic growth of positive solutions to a nonlocal elliptic blow-up system involving strong competition ☆

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Abstract

For a competition-diffusion system involving the fractional Laplacian of the form

$$-(-\Delta)^s u = uv^2, \quad -(-\Delta)^s v = vu^2, \quad u, v > 0 \text{ in } \mathbb{R}^N,$$

with $s \in (0, 1)$, we prove that the maximal asymptotic growth rate for its entire solutions is $2s$. Moreover, since we are able to construct symmetric solutions to the problem, when $N = 2$ with prescribed growth arbitrarily close to the critical one, we can conclude that the asymptotic bound found is optimal. Finally, we prove existence of genuinely higher dimensional solutions, when $N \geq 3$. Such problems arise, for example, as blow-ups of fractional reaction-diffusion systems when the interspecific competition rate tends to infinity.

MSC: primary 35J65; secondary 35B40, 35B44, 35R11, 81Q05, 82B10

Keywords: Fractional Laplacian; Spatial segregation; Strongly competing systems; Entire solutions

1. Introduction and main results

This paper deals with the existence and classification of positive entire solutions to polynomial systems involving the (possibly) s-fractional Laplacian of the following form:

$$-(-\Delta)^s u = uv^2, \quad -(-\Delta)^s v = vu^2, \quad u, v > 0 \text{ in } \mathbb{R}^N.$$

Such systems arise, for example, as blow-ups of fractional reaction-diffusion systems when the interspecific competition rate tends to infinity. In this framework, the existence and classification of entire solutions plays a key role in the

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asymptotic analysis (see, for instance, [15,17]). The case of standard diffusion ($s = 1$) has been intensively treated in the recent literature, also in connection with a De Giorgi-like conjecture about monotone solutions being one dimensional. In particular, a complete classification of solutions having linear growth (the lowest possible growth rate) has been given in [1,2,7–9,16,20]. On the other hand, when $s = 1$, positive solutions having arbitrarily large polynomial growth were discovered in [2] and with exponential growth in [14].

Competition-diffusion nonlinear systems with k-components involving the fractional Laplacian have been the object of a recent literature, starting with [18,19], where the authors provided asymptotic estimates for solutions to systems of the form

$$\begin{cases}(-\Delta)^s u_i = f_{i,\beta}(u_i) - \beta u_i \sum_{j\neq i} a_{ij}u_j^2, \quad i = 1,\dots,k,\\ u_i \in H^s(\mathbb{R}^N),\end{cases} \tag{1.1}$$

where $N \geq 2$, $a_{ij} = a_{ji} > 0$, when $\beta > 0$ (the competition parameter) goes to $+\infty$. Moreover we consider $f_{i,\beta}$ as continuous functions which are uniformly bounded on bounded sets with respect to β (see [18,19] for details). The fractional Laplacian is defined for every $s \in (0, 1)$ as

$$(-\Delta)^s u(x) = c(N,s)\ \mathrm{PV}\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,\mathrm{d}y.$$

In order to state our results, we adopt the approach of Caffarelli–Silvestre [5], and we see the fractional Laplacian as a Dirichlet-to-Neumann operator; that is, we consider the extension problem for (1.1). In other words, we study an auxiliary problem in the upper half space in one more dimension[1]; that is, letting $a := 1 - 2s$, for any $i = 1,\dots,k$ the localized version of (1.1),

$$\begin{cases}L_a u_i = 0, & \text{in } B_1^+ \subset \mathbb{R}^{N+1}_+,\\ \partial^a_y u_i = f_{i,\beta}(u_i) - \beta u_i \sum_{j\neq i} a_{ij}u_j^2, & \text{in } \partial^0 B_1^+ \subset \partial\mathbb{R}^{N+1}_+ = \mathbb{R}^N \times \{0\},\end{cases} \tag{1.2}$$

where the degenerate/singular elliptic operator L_a is defined as

$$-L_a u := \mathrm{div}(y^a \nabla u),$$

and the linear operator ∂^a_y is defined as

$$-\partial^a_y u := \lim_{y\to 0^+} y^a \frac{\partial u}{\partial y}.$$

The new problem (1.2) is equivalent to the original one when we deal with solutions in the energy space associated with the two operators. In fact a solution U to the extension problem is the extension of the correspondent solution u of the original nonlocal problem in the sense that $U(x, 0) = u(x)$. Let us remark that if $s = \frac{1}{2}$, then $a = 0$ and hence $L_0 = -\Delta$ and the boundary operator $-\partial^0_y$ becomes the usual normal derivative ∂_y. Moreover we remark that the extension problem has a variational nature in some weighted Sobolev spaces related to the Muckenhoupt A_2-weights (see for instance [10]). Hence, given $\Omega \subset \mathbb{R}^{N+1}_+$, we can introduce the Hilbert spaces

$$H^{1;a}(\Omega) := \left\{ u : \Omega \to \mathbb{R}\ :\ \int_\Omega y^a(|u|^2 + |\nabla u|^2) < +\infty \right\},$$

and

$$H^{1;a}_{loc}\left(\overline{\mathbb{R}^{N+1}_+}\right) := \left\{ u : \mathbb{R}^{N+1}_+ \to \mathbb{R}\ :\ \forall r > 0, u|_{B_r^+} \in H^{1;a}(B_r^+) \right\},$$

[1] Throughout this paper we assume the following notations: $z = (x, y)$ denotes a point in $\mathbb{R}^{N+1}_+$, with $x \in \partial\mathbb{R}^{N+1}_+ := \mathbb{R}^N$ and $y \in \mathbb{R}_+$. Moreover, $B_r^+(z_0) := B_r(z_0) \cap \mathbb{R}^{N+1}_+$ is the half ball, and its boundary is divided in the hemisphere $\partial^+ B_r^+(z_0) := \partial B_r^+(z_0) \cap \mathbb{R}^{N+1}_+$ and in the flat part $\partial^0 B_r^+(z_0) := \partial B_r^+(z_0) \setminus \partial^+ B_r^+(z_0)$. When the center of balls and spheres is omitted, then $z_0 = 0$.

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