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A family of degenerate elliptic operators: Maximum principle and its consequences ☆

Isabeau Birindelli [a,*], Giulio Galise [a,*], Hitoshi Ishii [b,*]

[a] *Dipartimento di Matematica "G. Castelnuovo", Sapienza Università di Roma, P.le Aldo Moro 2, I-00185 Roma, Italy*
[b] *Faculty of Education and Integrated Arts and Sciences, Waseda University, 1-6-1 Nishi-Waseda, Shinjuku, Tokyo 169-8050 Japan*

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Abstract

In this paper we investigate the validity and the consequences of the maximum principle for degenerate elliptic operators whose higher order term is the sum of k eigenvalues of the Hessian. In particular we shed some light on some very unusual phenomena due to the degeneracy of the operator. We prove moreover Lipschitz regularity results and boundary estimates under convexity assumptions on the domain. As a consequence we obtain the existence of solutions of the Dirichlet problem and of principal eigenfunctions.

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1. Introduction

In this paper we shall study solutions of Dirichlet problem for degenerate elliptic operators whose higher order term is given by some sort of "truncated Laplacian", i.e.

$$\mathcal{P}_k^-(D^2u) = \sum_{i=1}^{k} \lambda_i(D^2u) \quad \text{and} \quad \mathcal{P}_k^+(D^2u) = \sum_{i=N-k+1}^{N} \lambda_i(D^2u),$$

where $\lambda_1(D^2u) \le \lambda_2(D^2u) \le \cdots \le \lambda_N(D^2u)$ are the ordered eigenvalues of the Hessian of u. These operators have lately been investigated in various contexts e.g. [1,12–14,20,21,31,32]. We are interested in the case $N \ge 2$ and $k < N$

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* Corresponding authors.
E-mail addresses: isabeau@mat.uniroma1.it (I. Birindelli), galise@mat.uniroma1.it (G. Galise), hitoshi.ishii@waseda.jp (H. Ishii).

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since $\mathcal{P}_N^-(D^2u) = \mathcal{P}_N^+(D^2u) = \Delta u$. In the whole paper solutions are meant in the viscosity sense, see e.g. [16] and Definition 2.1.

Clearly, for any symmetric matrix X, $\mathcal{P}_k^+(X) = -\mathcal{P}_k^-(-X)$ hence we will mainly state the results for $\mathcal{P}_k^-$ with obvious equivalents when the operator $\mathcal{P}_k^+$ is considered. Such operators are positively homogeneous of degree one and degenerate elliptic.

In the following we propose to consider the Dirichlet problem

$$\begin{cases} \mathcal{P}_k^\pm(D^2u) + H(x,\nabla u) + \mu u = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ and the Hamiltonian $H \in C(\Omega \times \mathbb{R}^N; \mathbb{R})$ is assumed to satisfy the structure condition:

$$\exists b \in \mathbb{R}_+ \text{ s.t. } |H(x,\xi)| \le b|\xi| \quad \forall (x,\xi) \in \Omega \times \mathbb{R}^N. \tag{SC 1}$$

The prototype we have in mind is $H(x,\nabla u) = b(x)|\nabla u|$ or $H(x,\nabla u) = b(x)\cdot\nabla u$ with $b(x)$ a bounded continuous function in Ω.

In particular, in bounded domains Ω, we want to raise and partially answer the following questions, which are very intertwined:

(1) Under which conditions do the operators $\mathcal{P}_k^\pm(D^2u) + H(x,\nabla u) + \mu u$ satisfy the maximum principle, be it weak or strong?
(2) What are the regularity of the solutions of the Dirichlet problem?
(3) Do the principal eigenvalues and corresponding eigenfunctions exist?

In order to be more specific, let us describe what we call maximum or minimum principle in the sense of the *sign propagation property*.

Definition 1.1. F satisfies the maximum or weak maximum principle in Ω if

$$F[u] \ge 0 \text{ in } \Omega, \quad \limsup_{x\to\partial\Omega} u \le 0 \implies u \le 0 \text{ in } \Omega.$$

It satisfies the strong maximum principle if

$$F[u] \ge 0 \text{ in } \Omega, \quad u \le 0 \text{ in } \Omega \implies \text{either } u < 0 \text{ or } u \equiv 0.$$

Respectively, F satisfies the minimum or weak minimum principle in Ω if

$$F[u] \le 0 \text{ in } \Omega, \quad \liminf_{x\to\partial\Omega} u \ge 0 \implies u \ge 0 \text{ in } \Omega.$$

It satisfies the strong minimum principle if

$$F[u] \le 0 \text{ in } \Omega, \quad u \ge 0 \text{ in } \Omega \implies \text{either } u > 0 \text{ or } u \equiv 0.$$

Of course when F is odd then the notions of maximum and minimum principle are equivalent, but here we shall see that they differ quite a lot.

Just to give a flavor of the kind of results that we shall obtain, let us begin by saying that for any $k < N$, the Hopf Lemma, the Harnack inequality and the strong minimum principle do not hold in general for solutions of (1.1). On the other hand, if $bR \le k$, the weak minimum principle holds in any domain $\Omega \subset B_R$. For subsolutions, instead, the strong maximum principle will be a consequence of the Hopf Lemma. The condition $bR \le k$ has been shown to be optimal in a previous work of the second named author with Vitolo [18]. Other phenomena which are unusual with respect to the uniformly elliptic case will be described in subsection 4.2.

Historically, the maximum (or minimum) principle for degenerate elliptic operators has been mostly studied when the degeneracy depends on the points where the operator acts, e.g.

$$Lu = \mathrm{tr}(A(x)D^2u) \text{ with } A \ge 0$$

or