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Mean field games with congestion

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Abstract

We consider a class of systems of time dependent partial differential equations which arise in mean field type models with congestion. The systems couple a backward viscous Hamilton–Jacobi equation and a forward Kolmogorov equation both posed in $(0, T) \times (\mathbb{R}^N/\mathbb{Z}^N)$. Because of congestion and by contrast with simpler cases, the latter system can never be seen as the optimality conditions of an optimal control problem driven by a partial differential equation. The Hamiltonian vanishes as the density tends to $+\infty$ and may not even be defined in the regions where the density is zero. After giving a suitable definition of weak solutions, we prove the existence and uniqueness results of the latter under rather general assumptions. No restriction is made on the horizon T.

Keywords: Mean field games; Congestion models; Local coupling; Existence and uniqueness; Weak solutions

1. Introduction

Recently, an important research activity on mean field games (MFGs for short) has been initiated since the pioneering works [17–19] of Lasry and Lions: it aims at studying the asymptotic behavior of stochastic differential games (Nash equilibria) as the number n of agents tends to infinity. In these models, it is assumed that the agents are all identical and that an individual agent can hardly influence the outcome of the game. Moreover, each individual strategy is influenced by some averages of functions of the states of the other agents. In the limit when $n \to +\infty$, a given agent feels the presence of the others through the statistical distribution of the states. Since perturbations of a single agent's strategy does not influence the statistical states distribution, the latter acts as a parameter in the control problem to be solved by each agent. When the dynamics of the agents are independent stochastic processes, MFGs naturally lead to a coupled system of two partial differential equations (PDEs for short), a forward Kolmogorov or Fokker–Planck equation and a backward Hamilton–Jacobi–Bellman equation, see for example (1.1) below.

The theory of MFGs allows one to model congestion effects, i.e. situations in which the cost of displacement of the agents increases in those regions where the density is large. MFGs models including congestion were introduced and studied in [20]. A typical such model leads to the following system of PDEs:

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$$\begin{cases} -\partial_t u - \nu\Delta u + \frac{1}{\beta}\frac{|Du|^\beta}{(m+\mu)^\alpha} = F(t,x,m)\,, & (t,x)\in(0,T)\times\Omega \\ \partial_t m - \nu\Delta m - \operatorname{div}(m\frac{|Du|^{\beta-2}Du}{(m+\mu)^\alpha}) = 0\,, & (t,x)\in(0,T)\times\Omega \\ m(0,x)=m_0(x)\,,\ u(T,x)=G(x,m(T))\,, & x\in\Omega\,, \end{cases} \tag{1.1}$$

with $\nu > 0$, $\alpha > 0$, $\beta \in (1,2]$, $\mu \in \mathbb{R}$ with either $\mu > 0$ or $\mu = 0$. System (1.1) must generally be complemented with suitable boundary conditions on $(0,T)\times\partial\Omega$, but we will avoid the additional technical difficulties coming from the latter by focusing on the case when Ω is the flat torus, i.e. $\Omega = \mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$ and all the data are periodic.

Loosely speaking, (1.1) describes the optimization over a stochastic dynamics defined on a standard probability space $(\mathcal{X},\mathcal{F},\mathcal{F}_t,\mathbb{P})$

$$dX_t = w_t\,dt + \sqrt{2\nu}\,dB_t$$

where B_t is a N-dimensional Brownian motion, with a cost criterion given by

$$\inf_{w_t}\left[\mathbb{E}\int_0^T \left\{c_\beta\ (m_t+\mu)^\gamma\,|w_t|^{\frac{\beta}{\beta-1}} + F(t,X_t,m_t)\right\}dt + \mathbb{E}(G(X_T,m_T))\right], \tag{1.2}$$

where $\gamma = \frac{\alpha}{\beta-1}$ and c_β is a suitable normalization constant. In the viewpoint of the generic agent, $m_t = m(t,X_t)$ is meant to represent the distribution law of the states, however in the optimization process it is just, a priori, a given frozen density function. The mean field game equilibrium is next given through a fixed point scheme, by requiring, a posteriori, that m_t coincides with the law of the optimal process X_t.

In [20], P.-L. Lions put the stress on general structural conditions yielding the uniqueness for the following MFG systems with local coupling:

$$-\partial_t u - \nu\Delta u + H(t,x,m,Du) = F(t,x,m)\,,\quad (t,x)\in(0,T)\times\Omega \tag{1.3}$$

$$\partial_t m - \nu\Delta m - \operatorname{div}(mH_p(t,x,m,Du)) = 0\,,\quad (t,x)\in(0,T)\times\Omega \tag{1.4}$$

$$m(0,x)=m_0(x)\,,\ u(T,x)=G(x,m(T))\,,\quad x\in\Omega \tag{1.5}$$

namely that F and G be increasing w.r.t. m and that the following matrix be positive semidefinite:

$$\begin{pmatrix} -\frac{2}{m}\frac{\partial H}{\partial m}(t,x,m,p) & \frac{\partial}{\partial m}\nabla_p^T H(t,x,m,p) \\ \frac{\partial}{\partial m}\nabla_p H(t,x,m,p) & 2D^2_{p,p}H(t,x,m,p) \end{pmatrix} \geq 0 \tag{1.6}$$

for all $x\in\Omega$, $m>0$ and $p\in\mathbb{R}^N$. Since (1.1) is equivalent to (1.3)–(1.5) with $H(t,x,m,p) = \frac{1}{\beta}\frac{|p|^\beta}{(m+\mu)^\alpha}$, (1.6) becomes in this case

$$\alpha \leq \frac{4(\beta-1)}{\beta}. \tag{1.7}$$

In the present work, we will show that this hypothesis yields both the existence and the uniqueness of weak solutions.

Except for situations in which special tricks may be applied (stationary problems and quadratic Hamiltonian, see [13]), the existence of classical solutions of suitable generalizations of (1.1) seems difficult to obtain, because generally neither upper bounds on m nor strict positivity of m are known unless one restricts the growth conditions for the nonlinearities and assumes that the time horizon T is small, see [14,15] (see also [12] for the stationary case). Therefore, in order to get at a sufficiently general result, we aim at proving the existence and uniqueness of suitably defined weak solutions.

For MFG models without congestion, the first results on the existence of weak solutions were supplied in [19]. Besides, as already observed in [17–19], in the easiest cases, the system of PDEs can be seen as the optimality conditions of a problem of optimal control driven by a PDE: in such cases, a pair of primal-dual optimization problems can be introduced, leading to a suitable weak formulation for which there exists a unique solution, see [9], where possibly degenerate diffusions are dealt with. A striking fact is that in general, MFGs with congestion cannot be cast into an optimal control problem driven by a PDE, by contrast with simpler cases. For MFG systems (1.3)–(1.5) with H independent of m, a complete analysis is available in [24], which contains in particular new results on weak solutions of Fokker–Planck equations, and an answer to the delicate question of the uniqueness of weak solutions of