ARTICLE IN PRESS ANIHPC:2849

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Regularity of the Eikonal equation with two vanishing entropies

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Received 2 October 2016; received in revised form 6 June 2017; accepted 12 June 2017

Abstract

Let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected domain. The Eikonal equation $|\nabla u| = 1$ for a function $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ has very little regularity, examples with singularities of the gradient existing on a set of positive H^1 measure are trivial to construct. With the mild additional condition of two vanishing entropies we show ∇u is locally Lipschitz outside a locally finite set. Our condition is motivated by a well known problem in Calculus of Variations known as the Aviles–Giga problem. The two entropies we consider were introduced by Jin, Kohn [26], Ambrosio, DeLellis, Mantegazza [2] to study the Γ-limit of the Aviles–Giga functional. Formally if u satisfies the Eikonal equation and if

$$\nabla \cdot \left(\tilde{\Sigma}_{e_1e_2}(\nabla u^{\perp})\right) = 0 \text{ and } \nabla \cdot \left(\tilde{\Sigma}_{\epsilon_1\epsilon_2}(\nabla u^{\perp})\right) = 0 \text{ distributionally in } \Omega, \tag{1}$$

where $\tilde{\Sigma}_{e_1e_2}$ and $\tilde{\Sigma}_{\epsilon_1\epsilon_2}$ are the entropies introduced by Jin, Kohn [26], and Ambrosio, DeLellis, Mantegazza [2], then ∇u is locally Lipschitz continuous outside a locally finite set.

Condition (1) is motivated by the zero energy states of the Aviles–Giga functional. The zero energy states of the Aviles–Giga functional have been characterized by Jabin, Otto, Perthame [25]. Among other results they showed that if $\lim_{n\to\infty} I_{\epsilon_n}(u_n) = 0$ for some sequence $u_n \in W_0^{2,2}(\Omega)$ and $u = \lim_{n\to\infty} u_n$ then ∇u is Lipschitz continuous outside a finite set. This is essentially a corollary to their theorem that if u is a solution to the Eikonal equation $|\nabla u| = 1$ a.e. and if for every "entropy" Φ (in the sense of [18], Definition 1) function u satisfies $\nabla \cdot \left[\Phi(\nabla u^{\perp})\right] = 0$ distributionally in Ω then ∇u is locally Lipschitz continuous outside a locally finite set. In this paper we generalize this result in that we require only two entropies to vanish.

The method of proof is to transform any solution of the Eikonal equation satisfying (1) into a differential inclusion $DF \in K$ where $K \subset M^{2\times 2}$ is a connected compact set of matrices without Rank-1 connections. Equivalently this differential inclusion can be written as a constrained non-linear Beltrami equation. The set K is also non-elliptic in the sense of Sverak [32]. By use of this transformation and by utilizing ideas from the work on regularity of solutions of the Eikonal equation in fractional Sobolev space by Ignat [23], DeLellis, Ignat [15] as well as methods of Sverak [32], regularity is established.

MSC: 28A75

Keywords: Eikonal equation; Aviles Giga functional; Entropies; Non-linear Beltrami equation; Differential inclusions

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http://dx.doi.org/10.1016/j.anihpc.2017.06.002
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Please cite this article in press as: A. Lorent, G. Peng, Regularity of the Eikonal equation with two vanishing entropies, Ann. I. H. Poincaré – AN (2017), http://dx.doi.org/10.1016/j.anihpc.2017.06.002

1. Introduction

The Eikonal equation is a much studied equation whose more general form $|\nabla u| = f$ occurs in numerous areas of physics (geometric optics, wave propagation) and applied mathematics. Historically there has been great interest in first uniqueness and then subsequently regularity of the Eikonal equation. Uniqueness was largely resolved by the development of the theory of viscosity solutions [14], and subsequent regularity results have been established by a number of authors, [10,11]. Indeed, regularity and uniqueness of viscosity solutions of the Eikonal equation was one of the early triumphs that followed the development of the theory of viscosity solutions. Without additional hypotheses solutions of the Eikonal equation need have little regularity, it is easy to construct examples whose gradient is singular on a set of positive H^1 measure. One of the simplest Eikonal equations is

$$|\nabla u(x)| = 1 \text{ for } a.e.\ x \in \Omega, \tag{2}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded simply-connected domain. Our main theorem is a strong regularity result for solutions of equation (2) with an additional condition that is best described as having two vanishing entropies. The two entropies we consider were introduced into the study of the Aviles–Giga functional by Jin, Kohn [26], Ambrosio, DeLellis, Mantegazza [2], later works by DeSimone, Kohn, Müller, Otto [19], Jabine, Otto, Perthame [25] and Otto, DeLellis [16] characterized a wide class of entropies and used this characterization in a fundamental way to prove the strongest results known for the functional. In truth our main motivation also came from the Aviles–Giga functional and for this reason we will introduce it in some detail:

The Aviles–Giga functional is the second order functional

$$I_\epsilon(u) = \int_\Omega \frac{\left|1 - |\nabla u|^2\right|^2}{\epsilon} + \epsilon \left|\nabla^2 u\right|^2 \, dx$$

minimized over the space of functions $W_0^{2,2}(\Omega;\mathbb{R})$ or $W_0^{2,2}(\Omega;\mathbb{R}) \cap \{u : \nabla u(x) = \eta_x \text{ on } \partial\Omega\}$ where η_x is the inward pointing unit normal to $\partial\Omega$, where $\Omega \subset \mathbb{R}^2$ is a simply-connected Lipschitz domain. The Aviles–Giga functional I_ϵ forms a model for blistering and (in certain regimes) a model for liquid crystals [6,26,21]. In addition there is a closely related functional modeling thin magnetic films known as the *micromagnetics functional* [18,19,13,30,31,1,3]. For function $u \in W_0^{2,2}(\Omega)$ we refer to $I_\epsilon(u)$ as the *Aviles–Giga energy* of u. The Aviles–Giga functional is the most natural higher order generalization of the Modica–Mortola functional [28].

The biggest open problem in the study of the Aviles–Giga functional is the characterization of its Γ-limit, [6,7,26,2]. Given the structure of I_ϵ it is not a surprise that the conjectured limiting function class is a subspace of functions that satisfy the Eikonal equation (2). By analogy to the Modica–Mortola functional, it might be expected that the limiting function space is also a subspace of $\{v : \nabla v \in BV\}$ and the limiting energy is related to $\|D[\nabla u]\|$. However this is completely *false*; see the example following Theorem 3.9 of [2]. It is necessary to build a function class that is in a sense analogous to the function class $\{v : \nabla v \in BV\}$ that is tailored to the functional I_ϵ. This is done by introducing certain *entropies* on the space of solutions of the Eikonal equation. The divergence of these entropies will (by virtue of the structure of I_ϵ) form measures that in regular examples pick up the jump in the gradient ∇u. Specifically it can be shown [2,18] that if $u_n \in W_0^{2,2}(\Omega)$ with the property that $\limsup_{n\to\infty} I_{\epsilon_n}(u_n) < \infty$ then for some subsequence $\{n_k\}$ we have $u_{n_k} \overset{W^{1,3}(\Omega)}{\to} u$. This allows us to show that if the vector field $\Sigma_{\xi\eta} u$ is defined by

$$\Sigma_{\xi\eta} u := u_\xi \left(1 - u_\eta^2 - \frac{1}{3} u_\xi^2\right) \xi - u_\eta \left(1 - u_\xi^2 - \frac{1}{3} u_\eta^2\right) \eta, \tag{3}$$

(where u_ξ and u_η are the partial derivatives along ξ and η respectively) then $\nabla \cdot \left(\Sigma_{\xi\eta} u\right)$ is a measure. So instead of having that the gradient of the gradient is a measure (as would be the case if $u \in \{v : \nabla v \in BV\}$) we have that the divergence of a vector field made up of first order partial gradients is a measure, which “morally” is not that far away.

Following [2], we denote by (e_1, e_2) the canonical basis of $\mathbb{R}^2$, and by

$$\epsilon_1 := \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad \epsilon_2 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \tag{4}$$

the basis obtained from (e_1, e_2) under an anticlockwise rotation of $\frac{\pi}{4}$. It is straightforward to check that

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