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Korn inequalities for shells with zero Gaussian curvature

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Abstract

We consider shells with zero Gaussian curvature, namely shells with one principal curvature zero and the other one having a constant sign. Our particular interests are shells that are diffeomorphic to a circular cylindrical shell with zero principal longitudinal curvature and positive circumferential curvature, including, for example, cylindrical and conical shells with arbitrary convex cross sections. We prove that the best constant in the first Korn inequality scales like thickness to the power $3/2$ for a wide range of boundary conditions at the thin edges of the shell. Our methodology is to prove, for each of the three mutually orthogonal two-dimensional cross-sections of the shell, a "first-and-a-half Korn inequality"—a hybrid between the classical first and second Korn inequalities. These three two-dimensional inequalities assemble into a three-dimensional one, which, in turn, implies the asymptotically sharp first Korn inequality for the shell. This work is a part of mathematically rigorous analysis of extreme sensitivity of the buckling load of axially compressed cylindrical shells to shape imperfections.

Keywords: Korn's inequality; Shells; Nonlinear elasticity; Cones; Cylinders

1. Introduction

Classical first and second Korn inequalities have been known to play a central role in the theory of linear elasticity and recently they have found very important applications in the problems of buckling of slender structures [6,4,5]. Let us recall the classical first and second Korn inequalities, that actually date back to 1908, [8,9]. To that end we denote

$$\mathfrak{euc}(n) = \{\boldsymbol{u} : \mathbb{R}^n \to \mathbb{R}^n : \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b},\ \boldsymbol{A} \in \mathrm{Skew}(\mathbb{R}^n),\ \boldsymbol{b} \in \mathbb{R}^n\}$$

be the set of all infinitesimal motions, i.e., a Lie algebra of the group of all Euclidean transformations (rigid body motions). Let Ω be an open connected subset of $\mathbb{R}^n$ and $\boldsymbol{u} \in W^{1,2}(\Omega; \mathbb{R}^n)$. We denote[1] by $\mathrm{grad}\,\boldsymbol{u}$ and $(\mathrm{grad}\,\boldsymbol{u})_{\mathrm{sym}}$ the gradient and the symmetric part of the gradient, respectively, of a vector field $\boldsymbol{u}$. It is well-known that $(\mathrm{grad}\,\boldsymbol{u})_{\mathrm{sym}} = 0$

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[1] We reserve more streamlined notations $\nabla \boldsymbol{u}$ and $e(\boldsymbol{u})$ for "simplified" gradient and symmetrized gradient, respectively, that will be our main characters in the technical part of the paper.

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in Ω (in the sense of distributions) if and only if $\boldsymbol{u} \in \mathfrak{euc}(n)$. This is an immediate consequence of a simple observation (also very well-known) that all partial derivatives of the gradient $\boldsymbol{G} = \mathrm{grad}\,\boldsymbol{u}$ can be expressed as a linear combination of partial derivatives of the symmetric part of the gradient $\boldsymbol{E} = (\mathrm{grad}\,\boldsymbol{u})_{\mathrm{sym}}$:

$$\frac{\partial G_{jk}}{\partial x^i} = \frac{\partial E_{jk}}{\partial x^i} + \frac{\partial E_{ij}}{\partial x^k} - \frac{\partial E_{ik}}{\partial x^j}.$$

The classical first Korn inequality (e.g., as stated in [14]) quantifies this result by describing how large $(\mathrm{grad}\,\boldsymbol{u})_{\mathrm{sym}}$ must be if $\boldsymbol{u}$ lies in a closed subspace $V \subset W^{1,2}(\Omega; \mathbb{R}^n)$ that has trivial intersection with $\mathfrak{euc}(n)$. If Ω is a Lipschitz domain, then there exists a constant $K(\Omega, V)$, such that for every $\boldsymbol{u} \in V$

$$\|(\mathrm{grad}\,\boldsymbol{u})_{\mathrm{sym}}\|^2 \ge K(\Omega, V)\|\mathrm{grad}\,\boldsymbol{u}\|^2, \tag{1.1}$$

where $\|\cdot\|$ is the L^2-norm. The Korn constant $K(\Omega, V)$ measures the distance between the subspace V and $\mathfrak{euc}(n)$. The classical second Korn inequality asserts that the standard $W^{1,2}$ norm topology can be equivalently defined by replacing $\mathrm{grad}\,\boldsymbol{u}$ with $(\mathrm{grad}\,\boldsymbol{u})_{\mathrm{sym}}$:

$$\|\mathrm{grad}\,\boldsymbol{u}\|^2 \le C(\Omega)(\|(\mathrm{grad}\,\boldsymbol{u})_{\mathrm{sym}}\|^2 + \|\boldsymbol{u}\|^2), \quad \boldsymbol{u} \in W^{1,2}(\Omega; \mathbb{R}^n). \tag{1.2}$$

Originally, Korn inequalities were used to prove existence, uniqueness and well-posedness of boundary value problems of linear elasticity (see e.g., [11,1]). Nowadays, often, as in our particular case, it is the best Korn constant $K(\Omega, V)$ in the first Korn inequality that is of central importance (e.g., [2,12,13,15,16,10]). Specifically, we are interested in the asymptotic behavior of the Korn constant $K(\Omega_h, V_h)$ for shells with zero Gaussian curvature as a function of their thickness h for subspaces V_h of $W^{1,2}$ functions satisfying various boundary conditions at the thin edges of the shell. In [6,5] we have shown that $K(\Omega_h, V_h)$ represents an absolute lower bound on safe loads for any slender structure. For a classical circular cylindrical shell we have proved in [3] that $K(\Omega_h, V_h) \sim h^{3/2}$ for a broad class of boundary conditions at the thin edges of the shell.

The motivation for this work comes from the fact that the experimentally measured buckling loads of axially compressed cylindrical shells behave in a paradoxical way, dramatically disagreeing with predictions of classical shell theory. The universal consensus is that such behavior is due to the extreme sensitivity of shells to imperfections of shape and load. This study is a part of rigorous analytical investigation of the influence of shape on the structural behavior of cylindrical shells. It looks like (and this will be addressed in future work) the determining factor of the effect of shape imperfections is the Gaussian curvature of the shell's mid-surface as the Ansatzen in [17] suggest. In this paper we show that if the shell has a vanishing principal curvature (yielding zero Gaussian curvature), as in circular cylindrical shells, then the scaling of the Korn constant $K(\Omega_h, V_h)$ will remain unaffected, provided the nonzero principal curvature has a constant sign. Our analysis also shows that if both principal curvatures are zero on any open subset of the shell's mid-surface, then $K(\Omega_h, V_h) \sim h^2$. We conjecture that $K(\Omega_h, V_h) \sim h$ for shells of uniformly positive Gaussian curvature, while $K(\Omega_h, V_h) \sim h^{4/3}$ if the Gaussian curvature is negative on any open subset of the shell's middle surface, as suggested by test functions constructed in [17]. These conjectures will be addressed elsewhere.

The goal of this paper is to show that the tools developed in [3] for circular cylindrical shells, and extended and developed further in [7], possess enough flexibility to be applicable to a wide family of shells, and especially to cylindrically-shaped shells (the ones that have no boundary in one of the principal directions). The main idea is to first prove an inequality that is a hybrid between the first and second Korn inequalities (we call it "first-and-a-half Korn inequality" for this reason) by "assembling" it from its two-dimensional versions corresponding to cross-sections of the shell by curvilinear coordinate surfaces. The first Korn inequality is then a consequence of the first-and-a-half Korn inequality and an estimate on the normal component of $\boldsymbol{u} \in V_h$. We believe that this general methodology will work for broad classes of shells, even though the Gaussian curvature does affect both the final results and the validity of some of the technical steps in the proof, which must be suitably modified in each particular case. For example, the assumption of zero Gaussian curvature is essential for all main results in Section 3.

2. Preliminaries

Consider a shell whose mid-surface is of class C^2. Suppose z and θ are coordinates on the mid-surface of the shell, such that $z =$ constant and $\theta =$ constant are the lines of principal curvatures. Here θ will denote the circumferential

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