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Regularity estimates for quasilinear elliptic equations with variable growth involving measure data ☆

Sun-Sig Byun [a], Jihoon Ok [b], Jung-Tae Park [c,*]

[a] Department of Mathematical Sciences and Research Institute of Mathematics, Seoul National University, Seoul 08826, Republic of Korea
[b] Korea Institute for Advanced Study, Seoul 02455, Republic of Korea
[c] Department of Mathematical Sciences, Seoul National University, Seoul 08826, Republic of Korea

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Abstract

We investigate a quasilinear elliptic equation with variable growth in a bounded nonsmooth domain involving a signed Radon measure. We obtain an optimal global Calderón–Zygmund type estimate for such a measure data problem, by proving that the gradient of a very weak solution to the problem is as globally integrable as the first order maximal function of the associated measure, up to a correct power, under minimal regularity requirements on the nonlinearity, the variable exponent and the boundary of the domain.

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1. Introduction

There have been considerable theoretical advances in partial differential equations (PDEs) with variable exponent growth in recent years. The study of these problems has also become an important research field, and it represents various phenomena in applied sciences: for instance, electrorheological fluids [46], elasticity [52], flows in porous media [4], image restoration [18], thermo-rheological fluids [3], and magnetostatics [17].

In this paper, we consider the Dirichlet problem with measure data:

$$\begin{cases} -\operatorname{div} \mathbf{a}(Du, x) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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* Corresponding author.
E-mail addresses: byun@snu.ac.kr (S.-S. Byun), jihoonok@kias.re.kr (J. Ok), ppark00@snu.ac.kr (J.-T. Park).

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where Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 2$, with nonsmooth boundary $\partial\Omega$, and μ is a signed Radon measure on Ω with finite mass. We can assume, by extending μ by zero to $\mathbb{R}^n \setminus \Omega$, that μ is defined in $\mathbb{R}^n$ with $|\mu|(\Omega) = |\mu|(\mathbb{R}^n) < \infty$. The vector field $\mathbf{a} = \mathbf{a}(\xi, x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is differentiable in ξ and measurable in x, and it satisfies the following variable exponent growth and uniformly ellipticity conditions:

$$|\xi||D_\xi \mathbf{a}(\xi, x)| + |\mathbf{a}(\xi, x)| \leq \Lambda |\xi|^{p(x)-1}, \tag{1.2}$$

$$\lambda|\xi|^{p(x)-2}|\eta|^2 \leq \langle D_\xi \mathbf{a}(\xi, x)\eta, \eta\rangle, \tag{1.3}$$

for almost every $x \in \mathbb{R}^n$, every $\eta \in \mathbb{R}^n$, every $\xi \in \mathbb{R}^n \setminus \{0\}$, and appropriate constants λ, Λ. Here $D_\xi \mathbf{a}(\xi, x)$ is the Jacobian matrix of $\mathbf{a}$ with respect to ξ, $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$, and $p(\cdot)$ is a given continuous function in Ω satisfying

$$2 - \frac{1}{n} < \gamma_1 \leq p(\cdot) \leq \gamma_2 < \infty. \tag{1.4}$$

Note that (1.2) implies that $\mathbf{a}(0, x) = 0$ for $x \in \mathbb{R}^n$, and (1.3) yields the following monotonicity condition:

$$\langle \mathbf{a}(\xi_1, x) - \mathbf{a}(\xi_2, x), \xi_1 - \xi_2 \rangle \geq \begin{cases} \tilde{\lambda}\, |\xi_1 - \xi_2|^{p(x)} & \text{if} \quad p(x) \geq 2, \\ \tilde{\lambda}\left(|\xi_1|^2 + |\xi_2|^2\right)^{\frac{p(x)-2}{2}} |\xi_1 - \xi_2|^2 & \text{if} \quad 1 < p(x) < 2 \end{cases} \tag{1.5}$$

for all $x, \xi_1, \xi_2 \in \mathbb{R}^n$ and for some constant $\tilde{\lambda} = \tilde{\lambda}(n, \lambda, \gamma_1, \gamma_2) > 0$.

If $\gamma_1 > n$, then μ belongs to the dual space of $W_0^{1,p(\cdot)}(\Omega)$ as a consequence of Morrey's inequality and a duality argument, and so the existence and uniqueness of a weak solution u to (1.1) are well understood from the monotone operator theory, see for instance [49]. In this case, regularity estimates for (1.1) have been extensively studied, see for example [1,2,12,13,26,28,37]. For this reason, we only consider the case that $\gamma_1 \leq n$ for which a solution u of (1.1) in the distributional sense does not necessarily become a weak solution in $W_0^{1,p(\cdot)}(\Omega)$. In this respect, we need to consider a more general class of solutions below the duality exponent.

Definition 1.1. $u \in W_0^{1,1}(\Omega)$ is a *SOLA (Solution Obtained by Limits of Approximations)* to the problem (1.1) under the assumptions (1.2)–(1.4) if the vector field $\mathbf{a}(Du, x) \in L^1(\Omega, \mathbb{R}^n)$,

$$\int_\Omega \langle \mathbf{a}(Du, x), D\varphi \rangle \, dx = \int_\Omega \varphi \, d\mu$$

holds for all $\varphi \in C_c^\infty(\Omega)$, and moreover there exists a sequence of weak solutions $\{u_h\}_{h \geq 1} \in W_0^{1,p(\cdot)}(\Omega)$ of the Dirichlet problems

$$\begin{cases} -\text{div}\, \mathbf{a}(Du_h, x) = \mu_h & \text{in } \Omega, \\ u_h = 0 & \text{on } \partial\Omega \end{cases} \tag{1.6}$$

such that

$$u_h \to u \quad \text{in } W_0^{1,\max\{1, p(\cdot)-1\}}(\Omega) \quad \text{as } h \to \infty, \tag{1.7}$$

where $\{\mu_h\} \in L^\infty(\Omega)$ converges weakly to μ in the sense of measure and satisfies for each open set $V \subset \mathbb{R}^n$,

$$\limsup_{h \to \infty} |\mu_h|(V) \leq |\mu|(\overline{V}), \tag{1.8}$$

with μ_h defined in $\mathbb{R}^n$ by considering the zero extension to $\mathbb{R}^n$.

Throughout the paper, we consider $\mu_h := \mu * \phi_h$, where ϕ_h is the usual mollifier, and then $\mu_h \in C^\infty(\Omega)$ converges weakly to μ in the sense of measure satisfying (1.8) and the following uniform L^1-estimate:

$$\|\mu_h\|_{L^1(\Omega)} \leq |\mu|(\Omega). \tag{1.9}$$

With such μ_h, there exists a SOLA u of (1.1) belonging to $W_0^{1,q(\cdot)}(\Omega)$ for all $q(\cdot)$ with

$$1 \leq q(\cdot) < \min\left\{\frac{n(p(\cdot)-1)}{n-1}, p(\cdot)\right\}.$$