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Existence and non-existence of transition fronts for bistable and ignition reactions

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Abstract

We study reaction–diffusion equations in one spatial dimension and with general (space- or time-) inhomogeneous mixed bistable–ignition reactions. For those satisfying a simple quantitative hypothesis, we prove existence and uniqueness of transition fronts, as well as convergence of "typical" solutions to the unique transition front (the existence part even extends to mixed bistable–ignition–monostable reactions). These results also hold for all pure ignition reactions without the extra hypothesis, but not for all pure bistable reactions. In fact, we find examples of either spatially or temporally periodic pure bistable reactions (independent of the other space–time variable) for which we can prove non-existence of transition fronts. The latter are the first such examples in periodic media which are non-degenerate in a natural sense, and they also prove a conjecture from [7].

Keywords: Reaction–diffusion equations; Bistable reactions; Ignition reactions; Transition fronts

1. Introduction

We study reaction–diffusion equations

$$u_t = u_{xx} + f(x,u) \tag{1.1}$$

and

$$u_t = u_{xx} + f(t,u) \tag{1.2}$$

in one spatial dimension. These equations are used to model a host of natural processes such as combustion, population dynamics, pulse propagation in neural networks, or solidification dynamics. We will consider here the cases of either *spatially* (1.1) or *temporally* (1.2) *inhomogeneous mixed bistable–ignition reactions*. We are primarily interested in general (non-periodic) reactions, but our results are new even in the periodic case.

For homogeneous media, one usually considers bistable reactions to have $\tilde\theta \in (0,1)$ such that $f(0) = f(\tilde\theta) = f(1) = 0$, with $f < 0$ on $(0,\tilde\theta)$ and $f > 0$ on $(\tilde\theta, 1)$, while ignition reactions have $f = 0$ on $(0,\tilde\theta)$ and $f > 0$

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on $(\tilde{\theta}, 1)$. It is also standard to consider f non-increasing near 0 and 1 (and sometimes even $f'(1) < 0$, along with $f'(0) < 0$ for bistable f). One is then interested in solutions $0 \le u \le 1$ which transition between the (stable) equilibria $u \equiv 0$ and $u \equiv 1$, modeling invasions of one equilibrium of the relevant physical process by another. Typically these include solutions evolving from initial data which are *spark-like* (with $\lim_{|x|\to\infty} u(0,x) = 0$), or *front-like* (with $\lim_{x\to\infty} u(0,x) = 0$ and $\liminf_{x\to-\infty} u(0,x) > \tilde{\theta}$). It is customary to also assume $\int_0^1 f(u)du > 0$, so that solutions which are initially above some $\beta > \tilde{\theta}$ on a large enough β-dependent interval converge locally uniformly to 1 as $t \to \infty$ (i.e., they propagate). One is then interested in the nature of the transition from 0 to 1. (Note that the roles of 0 and 1 are reversed if $\int_0^1 f(u)du < 0$ for bistable f.)

The study of transitions between equilibria of reaction–diffusion equations has seen a lot of activity since the seminal papers of Kolmogorov, Petrovskii, Piskunov [16] and Fisher [13] (who studied homogeneous reactions). We are here interested in this question for f which also depends on x or t, and we will also relax the requirement for a single sign change of $f(x,\cdot)$ or $f(t,\cdot)$ in $(0,1)$. We will therefore assume the following hypothesis. Let us consider only (1.1) for the time being; (1.2) will be treated afterwards.

Hypothesis (H): f is Lipschitz with constant $K \ge 1$,

$$f(x,0) = f(x,1) = 0 \qquad \text{for } x \in \mathbb{R}, \tag{1.3}$$

and there is $\theta > 0$ such that for each $x \in \mathbb{R}$, f is non-increasing in u on $[0,\theta]$ and on $[1-\theta, 1]$. Moreover, there are $0 < \theta_1 \le \theta_0 < 1$ and Lipschitz functions $f_0, f_1 : [0,1] \to \mathbb{R}$ with $f_0 \le f_1$,

$$\begin{aligned} &f_0(0) = f_0(1) = f_1(0) = f_1(1) = 0, \\ &f_0 \le 0 \text{ on } (0,\theta_0) \qquad \text{and} \qquad f_0 > 0 \text{ on } (\theta_0, 1), \\ &f_1 \le 0 \text{ on } (0,\theta_1) \qquad \text{and} \qquad f_1 > 0 \text{ on } (\theta_1, 1), \end{aligned}$$

$$\int_0^1 f_0(u)du > 0, \tag{1.4}$$

such that

$$f_0(u) \le f(x,u) \le f_1(u) \qquad \text{for } (x,u) \in \mathbb{R} \times [0,1].$$

Definition 1.1.

(i) We call any f satisfying (H) a *BI reaction* (i.e., bistable–ignition).

(ii) If f is a BI reaction and $f_1 < 0$ on $(0,\theta_1)$, then f is a *bistable reaction*. If there is also an increasing function $\gamma : [0,\infty) \to [0,\infty)$ and for each $x \in \mathbb{R}$ there is $\tilde{\theta}_x \in [\theta_1, \theta_0]$ such that

$$\operatorname{sgn}(u - \tilde{\theta}_x) f(x,u) \ge \gamma\big(\operatorname{dist}(u, \{0, \tilde{\theta}_x, 1\})\big)$$

for $u \in [0,1]$, then f is a *pure bistable reaction*.

(iii) If f is a BI reaction and $f_0 = 0$ on $(0,\theta_0)$, then f is an *ignition reaction*. If there are also γ and $\tilde{\theta}_x$ as in (ii) such that now $f(x,u) = 0$ for $u \in [0,\tilde{\theta}_x]$ and

$$f(x,u) \ge \gamma\big(\operatorname{dist}(u, \{\tilde{\theta}_x, 1\})\big)$$

for $u \in [\tilde{\theta}_x, 1]$, then f is a *pure ignition reaction*.

Remark. We note that if instead $\theta = 0 = \theta_1$ in (H), then f is a *mixed bistable–ignition–monostable reaction*, and it is a *pure monostable reaction* if (iii) above holds with $\tilde{\theta}_x \equiv 0$.

Let us now briefly review some of the relevant literature for bistable and ignition reactions in one dimension (their mixtures, allowed here, may not have been studied before). In these papers, (1.4) need not always be assumed for bistable reactions and other hypotheses may be included. There is also a large body of work on monostable reactions

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