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Damping of particles interacting with a vibrating medium

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Abstract

We investigate the large time behavior of the solutions of a Vlasov–Fokker–Planck equation where particles are subjected to a confining external potential and a self-consistent potential intended to describe the interaction of the particles with their environment. The environment is seen as a medium vibrating in a direction transverse to particles' motion. We identify equilibrium states of the model and justify the asymptotic trend to equilibrium. The analysis relies on hypocoercivity techniques.

MSC: 82C70; 70F45; 37K05; 74A25

Keywords: Vlasov–Fokker–Planck equations; Interacting particles; Inelastic Lorentz gas; Relaxation to equilibrium; Hypocoercivity

1. Introduction

This work concerns the long-time behavior of the solution of the Vlasov equation

$$\partial_t F + v \cdot \nabla_x F - \nabla_x (V + \Phi) \cdot \nabla_v F = \gamma \nabla_v \cdot (vF + \nabla_v F), \qquad t \geq 0,\ x \in \mathbb{R}^d,\ v \in \mathbb{R}^d, \tag{1}$$

where Φ is self-consistently defined by the relations

$$\begin{cases} \Phi(t,x) = \displaystyle\int_{\mathbb{R}^d \times \mathbb{R}^n} \sigma_1(x-y)\sigma_2(z)\Psi(t,y,z)\,\mathrm{d}y\,\mathrm{d}z, & t \geq 0,\ x \in \mathbb{R}^d, \\ \left(\partial_{tt}^2 \Psi - c^2 \Delta_z \Psi\right)(t,x,z) = -\sigma_2(z) \displaystyle\int_{\mathbb{R}^d} \sigma_1(x-y)\rho(t,y)\,\mathrm{d}y, & t \geq 0,\ x \in \mathbb{R}^d,\ z \in \mathbb{R}^n, \\ \text{with } \rho(t,x) = \displaystyle\int_{\mathbb{R}^d} F(t,x,v)\,\mathrm{d}v. \end{cases} \tag{2}$$

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The system is complemented with the initial data

$$F(0,x,v) = F_0(x,v), \qquad \Psi(0,x,z) = \Psi_0(x,z), \qquad \partial_t \Psi(0,x,z) = \Psi_1(x,z). \tag{3}$$

The parameters of the problem are set as follows

- $c > 0$,
- $\sigma_1 : \mathbb{R}^d \to [0,\infty)$ and $\sigma_2 : \mathbb{R}^n \to [0,\infty)$ are radially symmetric C^∞ compactly supported functions,
- $V : \mathbb{R}^d \to \mathbb{R}$ is an external *confining* potential:

$$V \in C^0 \cap W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^d), \qquad \lim_{|x|\to\infty} V(x) = \infty.$$

We will make the technical assumptions precise later on. A crucial role in the analysis will be played by the following entropy dissipation property

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{ \int\limits_{\mathbb{R}^d\times\mathbb{R}^d} \Big(F\frac{v^2}{2} + F(V+\Phi) + F\ln(F)\Big)\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2}\int\limits_{\mathbb{R}^d\times\mathbb{R}^n} (|\partial_t\Psi|^2 + c^2|\nabla_z\Psi|^2)\,\mathrm{d}z\,\mathrm{d}x\Big\} \\ = -\gamma \int\limits_{\mathbb{R}^d\times\mathbb{R}^d} |2\nabla_v\sqrt{F} + v\sqrt{F}|^2\,\mathrm{d}v\,\mathrm{d}x \le 0.\end{aligned} \tag{4}$$

The investigation of this problem is motivated by the work of S. De Bièvre and L. Bruneau [6] where a related model was introduced to describe the evolution of a single particle interacting with its environment. In [6] the particle is classically described by the pair position/velocity $(q(t), \dot q(t))$, and the dynamics is governed by

$$\begin{cases} \ddot q(t) = -\nabla V(q(t)) - \displaystyle\int\limits_{\mathbb{R}^d\times\mathbb{R}^n} \sigma_1(q(t)-y)\,\sigma_2(z)\,\nabla_x\Psi(t,y,z)\,\mathrm{d}y\,\mathrm{d}z, \\ \partial^2_{tt}\Psi(t,x,z) - c^2\Delta_z\Psi(t,x,z) = -\sigma_2(z)\sigma_1(x-q(t)), \qquad x\in\mathbb{R}^d,\ z\in\mathbb{R}^n. \end{cases} \tag{5}$$

Such single particle description can be retrieved by setting $F(t,x,v) = \delta(x = q(t)) \otimes \delta(v = \dot q(t))$ in (1), with $\gamma = 0$. The dynamics can be thought of as if membranes continuously distributed transversely to the direction of the particle's motion — $z \in \mathbb{R}^n$ being transverse to $x \in \mathbb{R}^d$ — were activated by the passage of the particle, see Fig. 1 in [6]. The evolution of the system is, therefore, driven by energy exchanges between the particle and the membranes. We remark that the coupling between the particle and the membranes is embodied into the *product* $\sigma_1(x)\sigma_2(z)$, which appears symmetrically in the two equations of (5). This is crucial to establish Hamiltonian properties of (5) and its counterpart for the kinetic model, namely relation (4). The system is presented as a "dynamical Lorentz gas" and one is interested in asymptotic properties of the dynamics. This question has been further investigated in a series of papers by S. De Bièvre and his collaborators [1,9–11,22], that contains both theoretical results and convincing numerical experiments. On the one hand, the system has certain dissipative features: under certain circumstances (roughly speaking, $n = 3$ and c large enough) the particle energy can be dissipated in the membranes, and the environment behaves like a friction force on the particle. In particular, when V is a confining potential with a (non-degenerate) minimum at q_0, then the particle stops at the location q_0 as time goes to ∞, see [6, Section 5, Theorem 4]. On the other hand, in [1,10] an approximated model is proposed, together with an interpretation of the dynamics in terms of random walk. This simplified framework permits to justify the approach to thermal equilibrium: the particle's momentum distribution is driven to a Maxwell–Boltzmann distribution.

We wish to revisit these questions in the framework of kinetic equations, where the description by the position/velocity pair is replaced by (1) when considering a distribution of particles in phase space $F(t,x,v) \ge 0$. More precisely, in the case $\gamma = 0$, $\int_A F(t,x,v)\,\mathrm{d}v\,\mathrm{d}x$ can be interpreted as the probability $\mathbb{P}\big((q(t),\dot q(t)) \in A\big)$ when the initial state of the particle is distributed according to F_0. The analysis of existence and uniqueness of weak solutions for the non-linearly coupled problem (1)–(2), with $\gamma = 0$, was established in [8], where it was also shown that a certain physical regime drives the solutions of (1)–(2) to solutions of the attractive Vlasov–Poisson system. It is likely that this approach can be combined to the analysis of the smoothing effect of the Fokker–Planck operator in [4,5] in order to investigate the well-posedness of the problem when $\gamma > 0$. We will not elaborate more on this issue in this paper

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