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# A quantitative theory for the continuity equation

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**Abstract**

In this work, we provide stability estimates for the continuity equation with Sobolev vector fields. The results are inferred from contraction estimates for certain logarithmic Kantorovich–Rubinstein distances. As a by-product, we obtain a new proof of uniqueness in the DiPerna–Lions setting. The novelty in the proof lies in the fact that it is not based on the theory of renormalized solutions.


*Keywords:* Continuity equations; Stability estimates; Logarithmic cost functions; Contraction estimates; Uniqueness

## 1. Introduction

When $u : [0,T] \times \mathbf{R}^d \to \mathbf{R}^d$, $f : [0,T] \times \mathbf{R}^d \to \mathbf{R}$ and $\bar\rho : \mathbf{R}^d \to \mathbf{R}$ are smooth functions, the solution of the Cauchy problem for the continuity equation

$$\begin{cases} \partial_t \rho + \nabla \cdot (u\rho) = f, \\ \rho(0,\cdot) = \bar\rho \end{cases} \tag{1}$$

is found by the method of characteristics: If we denote by $\phi : [0,T] \times \mathbf{R}^d \to \mathbf{R}^d$ the flow of the vector field $u$, i.e.,

$$\begin{cases} \partial_t \phi(t,x) = u(t,\phi(t,x)), \\ \phi(0,x) = x, \end{cases} \tag{2}$$

for any $(t,x) \in [0,T] \times \mathbf{R}^d$, then the solution to (1) is given by the formula

$$\rho(t,\phi(t,x)) \det \nabla\phi(t,x) = \bar\rho(x) + \int_0^t f(s,\phi(s,x)) \det \nabla\phi(s,x)\, ds. \tag{3}$$

In the non-smooth setting, solutions have to be defined in the sense of distributions. A complete theory of distributional solutions, including existence, uniqueness and stability properties, is provided in the seminal works of DiPerna and Lions [13] and Ambrosio [2].

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The approach of DiPerna, Lions and Ambrosio relies on the theory of renormalized solutions. Roughly speaking, renormalized solutions are distributional solutions to which the chain rule applies in the sense that, for every suitable $\beta \in C^1(\mathbf{R})$, $\beta(\rho)$ solves the continuity equation with source $\beta'(\rho)f + (\nabla \cdot u)(\beta(\rho) - \rho\beta'(\rho))$ and initial datum $\beta(\bar{\rho})$. Whether distributional solutions are renormalized solutions depends on the regularity of $u$. It has been proved in the original paper by DiPerna and Lions [13] that this is true under the condition that $u \in L^1(W^{1,1})$ and $\nabla \cdot u \in L^1(L^\infty)$. DiPerna and Lions furthermore show that the latter can not be relaxed, in the sense that there are (stationary) vector fields in $W^{1,p}$ for any $p < \infty$ whose divergence is unbounded and that do not possess this renormalization property. Likewise, the authors construct solutions to the continuity equation with $u \in W^{s,1}$ for any $s < 1$ (and $\nabla \cdot u = 0$) that are not renormalized. In [2], Ambrosio extends DiPerna's and Lions's results to vector fields $u \in L^1(BV)$. A counterexample in the non-$BV$ setting is provided by Depauw [9].

The merit of renormalization theory relies on a simple proof of uniqueness and stability. For instance, if $\eta$ denotes the difference of two solutions to the Cauchy problem (1), the choice $\beta(z) = z^2$ yields

$$\partial_t \eta^2 + \nabla \cdot \left(u\eta^2\right) = -(\nabla \cdot u)\eta^2,$$

and thus, integration in space and a Gronwall argument shows that

$$\|\eta\|_{L^\infty(L^2)} \le \|\bar{\eta}\|_{L^2} \exp^{\frac{1}{2}}\left(\|\nabla \cdot u\|_{L^1(L^\infty)}\right).$$

Thus, if the initial datum $\bar{\eta}$ is zero, then $\eta$ vanishes identically. For a recent review on the well-posedness theories for the continuity equation (1) and the related ordinary differential equation (2), we refer the reader to the lecture notes [3].

Renormalization theory is also powerful as it applies to a fairly broad class of transport or kinetic equations, e.g., [10,12,11]. What the theory does not provide are stability estimates and bounds on the mixing or unmixing efficiencies in terms of the regularity of the advecting vector field. Such estimates, however, attracted much attention recently. For instance, in [18,23,16], the continuity equation is considered as model for mixing of tracer particles by a viscous fluid flow. An important question in engineering applications is how well tracers can be mixed under a constraint on the advecting velocity field. Typically, one is interested in optimal mixing rates in terms of the kinetic energy $\|u\|_{L^2}$ or, more importantly, the viscous dissipation $\|\nabla u\|_{L^2}$. The works [23,16] provide lower bounds on the rate of exponential decay of the $H^{-1}$ norm by $\|\nabla u\|_{L^1(L^p)}$. Optimality of these bounds is proved in [1,27].

The goal of the present work is to establish stability estimates for continuity equations with Sobolev vector fields that allow for variations of vector field, source, and initial datum. We demonstrate the strength of these estimates by providing a new proof of uniqueness of distributional solutions. Opposed to the theory of DiPerna, Lions, and Ambrosio, our approach does not rely on renormalized solutions. Instead, we obtain uniqueness from a contraction estimate under suitable integrability assumptions on the solutions.

Our approach is motivated by a related work by Crippa and De Lellis for the ordinary differential equation (2). In [7], the authors derive simple stability estimates for suitably generalized flows, so-called regular Lagrangian flows, in the case of Sobolev vector fields $u$. These estimates allow for a direct proof of well-posedness. Prior to the work of Crippa and De Lellis, uniqueness and stability were obtained quite indirectly and exploited the connection to the continuity equation (1) and the transport equation

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = f, \\ \rho(0, \cdot) = \bar{\rho} \end{cases}$$

via the method of characteristics, cf. [13,2]. Crippa's and De Lellis's approach has been partially extended to the $BV$ setting later by Jabin [17] and Hauray and Le Bris [15].

Focusing on the case $p > 1$, Crippa and De Lellis prove that any two solutions $\phi$ and $\tilde{\phi}$ of (2) satisfy estimates of the type

$$\sup_{t \in (0,T)} \int \log\left(\frac{|\phi(t,x) - \tilde{\phi}(t,x)|}{\delta} + 1\right) dx \lesssim \|\nabla u\|_{L^1(L^p)}, \tag{4}$$

*uniformly* in $\delta > 0$. That means, trajectories can only vary in a tube with diameter of order $\delta$. As $\delta \to 0$, this tube shrinks to a single curve, which proves uniqueness. Crippa's and De Lellis's logarithmic estimates generalize the well-known estimate valid for flows of Lipschitz vector fields

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