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A global weak solution of the Dirac-harmonic map flow ☆

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Abstract

We show the existence of a global weak solution of the heat flow for Dirac-harmonic maps from compact Riemann surfaces with boundary when the energy of the initial map and the L^2-norm of the boundary values of the spinor are sufficiently small. Dirac-harmonic maps couple a second order harmonic map type system with a first-order Dirac type system. The heat flow that has been introduced in [9] and that we investigate here is novel insofar as we only make the second order part parabolic, but carry the first order part along the resulting flow as an elliptic constraint. Of course, since the equations are coupled, both parts then change along the flow.

The solution is unique and regular with the exception of at most finitely many singular times. We also discuss the behavior at the singularities of the flow.

As an application, we deduce some existence results for Dirac-harmonic maps. Since we may impose nontrivial boundary conditions also for the spinor part, in the limit, we shall obtain Dirac-harmonic maps with nontrivial spinor part.

MSC: 53C43; 58E20

Keywords: Dirac-harmonic map; Dirac-harmonic flow; Blow-up; Dirichlet boundary; Chiral boundary; Singularity

1. Introduction

Motivated by the nonlinear supersymmetric sigma model from quantum field theory, Dirac-harmonic maps are critical points of an energy functional that couples maps with spinor fields. They were introduced in Chen–Jost–Li–Wang [7,8]. This subject generalizes the theory of harmonic maps and harmonic spinors. The particular structure of the coupling which comes from the nonlinear supersymmetric sigma model is crucial for their subtle geometric and analytical properties. This structure needs to be very carefully exploited and combined with some of the most power-

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ful and advanced techniques and results in geometric analysis in order to derive regularity, existence and uniqueness results. This is the context of the present paper. We shall discuss and analyze a parabolic version of the model. Since the action functional contains a field term which is quadratic in the field derivatives and a spinor term which is linear in the Dirac derivative of that spinor, the solutions of the resulting Euler–Lagrange system, the Dirac-harmonic maps, consist of a second order harmonic map type system and a first-order Dirac type system. Since the map and the spinor are coupled in the action functional, the resulting Euler–Lagrange equations are likewise coupled. In order to treat this somewhat unusual situation, we work with a heat flow that was introduced in [9]. This heat flow is non-standard insofar as we only make the second order part parabolic, but carry the first order part along the resulting flow as an elliptic constraint. Of course, since the equations are coupled, both parts then change along the flow. We shall show the existence of a unique global weak solution under some smallness assumptions on the initial data. As is to be expected for such problems, we encounter the possibility of finite time blow-up, and therefore the weak solution in general will not be strong. But at least, it can be continued across such a singularity as a weak solution.

1.1. The Dirac-harmonic variational problem

In order to discuss our results in more detail, we now need to become more technical. Let us first present the Dirac-harmonic model, which this paper is about. Let (M, g) be a Riemann surface with a fixed spin structure, ΣM the spinor bundle over M and $\langle\cdot,\cdot\rangle_{\Sigma M}$ the metric on ΣM. Choosing a local orthonormal basis $e_\alpha, \alpha = 1, 2$ on M, the usual Dirac operator is defined as $\partial\!\!\!/ := e_\alpha \cdot \nabla_{e_\alpha}$, where ∇ is the spin connection on ΣM and $\cdot$ is the Clifford multiplication. This multiplication is skew-adjoint:

$$\langle X \cdot \psi, \varphi\rangle_{\Sigma M} = -\langle \psi, X \cdot \varphi\rangle_{\Sigma M}$$

for any $X \in \Gamma(TM)$, $\psi,\ \varphi \in \Gamma(\Sigma M)$.

The usual Dirac operator $\partial\!\!\!/$ on a surface can be seen as the (doubled) Cauchy–Riemann operator. Consider $\mathbb{R}^2$ with the Euclidean metric $dx^2 + dy^2$. Let $e_1 = \frac{\partial}{\partial x}$ and $e_2 = \frac{\partial}{\partial y}$ be the standard orthonormal frame. A spinor field on $\mathbb{R}^2$ is simply a map $\psi : \mathbb{R}^2 \to \Sigma\mathbb{R}^2 = \mathbb{C}^2$, and the action of e_1 and e_2 on spinors can be identified with multiplication with matrices

$$e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

If $\psi := \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{C}^2$ is a spin field, then the Dirac operator is

$$\partial\!\!\!/\psi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial\psi_1}{\partial x} \\ \frac{\partial\psi_2}{\partial x} \end{pmatrix} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial\psi_1}{\partial y} \\ \frac{\partial\psi_2}{\partial y} \end{pmatrix} = 2\begin{pmatrix} \frac{\partial\psi_2}{\partial \bar{z}} \\ -\frac{\partial\psi_1}{\partial z} \end{pmatrix}, \tag{1.1}$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}).$$

For more details on spin geometry and Dirac operators, we can refer to [18].

Let ϕ be a smooth map from M to some compact Riemannian manifold (N, h) with dimension $n \geq 2$. If $\phi^{-1}TN$ is the pull-back bundle of TN by ϕ, we get the twisted bundle $\Sigma M \otimes \phi^{-1}TN$. Naturally, there is a metric $\langle\cdot,\cdot\rangle_{\Sigma M\otimes\phi^{-1}TN}$ on $\Sigma M \otimes \phi^{-1}TN$ which is induced from the metrics on ΣM and $\phi^{-1}TN$. Also we have a natural connection $\widetilde{\nabla}$ on $\Sigma M \otimes \phi^{-1}TN$ which is induced from the connections on ΣM and $\phi^{-1}TN$. Let ψ be a section of the bundle $\Sigma M \otimes \phi^{-1}TN$. In local coordinates, it can be written as

$$\psi = \psi^i \otimes \partial_{y^i}(\phi),$$

where each ψ^i is a standard spinor on M and ∂_{y^i} is the natural local basis of TN. Then $\widetilde{\nabla}$ becomes

$$\widetilde{\nabla}\psi = \nabla\psi^i \otimes \partial_{y^i}(\phi) + (\Gamma^i_{jk}\nabla\phi^j)\psi^k \otimes \partial_{y^i}(\phi), \tag{1.2}$$

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