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Plates with incompatible prestrain of high order

Marta Lewicka [a], Annie Raoult [b,*], Diego Ricciotti [a]

[a] *University of Pittsburgh, Department of Mathematics, 301 Thackeray Hall, Pittsburgh, PA 15260, USA*
[b] *Laboratoire MAP5, Université Paris Descartes & CNRS, Sorbonne Paris Cité, France*

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Abstract

We study the elastic behaviour of incompatibly prestrained thin plates of thickness h whose internal energy E^h is governed by an imposed three-dimensional smooth Riemann metric G only depending on the variable in the midsurface ω. It is already known that $h^{-2}\inf E^h$ converges to a finite value c when the metric G restricted to the midsurface has a sufficiently regular immersion, namely $W^{2,2}(\omega,\mathbb{R}^3)$. The obtained limit model generalizes the bending (Kirhchoff) model of Euclidean elasticity. In the present paper, we deal with the case when c equals 0. Then, equivalently, three independent entries of the three-dimensional Riemann curvature tensor associated with G are null. We prove that, in such regime, necessarily $\inf E^h \le Ch^4$. We identify the Γ-limit of the scaled energies $h^{-4}E^h$ and show that it consists of a von Kármán-like energy. The unknowns in this energy are the first order incremental displacements with respect to the deformation defined by the bending model and the second order tangential strains. In addition, we prove that when $\inf h^{-4}E^h \to 0$, then G is realizable and hence $\min E^h = 0$ for every h.

Résumé

On s'intéresse au comportement de structures minces d'épaisseur h dont l'énergie interne E^h est régie par une métrique riemannienne tridimensionnelle G imposée, constante dans l'épaisseur, n'admettant pas nécessairement d'immersion isométrique. On sait que lorsque la restriction de G à la surface moyenne ω possède une immersion isométrique suffisamment régulière, c'est-à-dire appartenant à $W^{2,2}(\omega,\mathbb{R}^3)$, alors $h^{-2}\inf E^h$ admet une limite finie c quand h tend vers 0. Le modèle limite correspondant généralise le modèle de flexion non linéaire, classique pour la métrique euclidienne. Nous nous plaçons ici dans le cas où c vaut 0, ce qui équivaut à la nullité de trois des six coeffiecients du tenseur de courbure associé à G. Nous montrons qu'alors $\inf E^h \le Ch^4$. Nous identifions la Γ-limite de $h^{-4}E^h$ et montrons qu'elle généralise l'énergie de von Kármán. Elle s'exprime en fonction des déplacements incrémentaux par rapport à la surface définie par le modèle de flexion et de déformations tangentielles généralisées. De plus, nous montrons que l'infimum de ce modèle limite à l'ordre 4 n'est nul que si G admet une immersion isométrique, auquel cas $\min E^h = 0$ pour tout h.

Keywords: Non-Euclidean plates; Prestrained energy; Nonlinear elasticity; Calculus of variations

* Corresponding author.
E-mail addresses: lewicka@pitt.edu (M. Lewicka), annie.raoult@parisdescartes.fr (A. Raoult), DIR17@pitt.edu (D. Ricciotti).

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1. Introduction

The purpose of this paper is to study the elastic behaviour of pre-stressed thin plates Ω^h, characterized by non-immersable Riemannian metrics G on their reference configurations. To such metrics (and the prestrain they generate), we refer as "incompatible"; the incompatibility is measured by infimizing the energy E^h given below, sometimes called the "non-Euclidean" elastic energy. We will be concerned with the regimes of curvatures of G resulting in the incompatibility of "high order". By this we mean that $\inf E^h \sim h^\beta$ as the plate's thickness h goes to 0, and that the scaling exponent β satisfies: $\beta > 2$.

In paper [5] we analyzed the scaling $\inf E^h \sim h^2$ and proved that it only occurs when the metric $G_{2\times 2}$ on the mid-plate has an isometric immersion in $\mathbb{R}^3$ with the regularity $W^{2,2}$ and when, at the same time, the three tangential Riemann curvatures of G do not vanish identically. The two-dimensional limiting energy, obtained from the sequence $h^{-2}E^h$ via Γ-convergence, as $h \to 0$, was an extension of the classical nonlinear bending energy.

In the present paper we assume that:

$$h^{-2} \inf E^h \to 0 \tag{1.1}$$

and prove that the only nontrivial two-dimensional limiting theory in this regime is a von Kármán-like energy, valid when $\inf E^h \sim h^4$. It further turns out that this scaling is automatically implied by (1.1) and $\inf E^h \neq 0$. Indeed, we show that (1.1) implies $h^{-4} \inf E^h \leq C$, and that $h^{-4} \inf E^h \to 0$ if and only if G is immersable, in which case $\min E^h = 0$ for all h.

Let us observe that this scale separation is different from the findings of [29] valid in the Euclidean case of $G = \mathrm{Id}_3$, where the possible limiting energies are distinguished by the scaling of the applied forces $f^h \sim h^\alpha$. In that context, three distinct limiting theories have been obtained for $\inf E^h \sim h^\beta$ with $\beta > 2$ (corresponding to $\alpha > 2$). Namely: $\beta \in (2, 4)$ yielded the linearized bending model subject to a nonlinear constraint on the limiting displacements; $\beta = 4$ yielded the classical von Kármán model; and $\beta > 4$ corresponded to the linear elasticity.

The present results differ as well from the higher order hierarchy of scalings and the elastic theories of shells, as derived through an asymptotic expansion in [46]. These differences are due to the fact that while the magnitude of external forces is adjustable at will, it is not so for the six sectional curvatures (together with their covariant derivatives) of a given metric G. As we show, the six curvatures of $G = G(x')$ depending only on the mid-plate variables fall into two categories: including or excluding the thin direction variable. Then, the simultaneous vanishing of curvatures in the first category or in both categories correspond to the two scenarios at hand in terms of the scaling of $\inf E^h$.

1.1. Some background in dimension reduction for thin structures

Early attempts for replacing the three-dimensional model of a thin elastic structure with planar mid-surface at rest, by a two-dimensional model, were based on *a priori* simplifying assumptions on the deformations and on the stresses. Later, the natural idea of using the thickness as a small parameter and of establishing a limit model was largely explored; we refer in particular to the works by Ciarlet and Destuynder who set the method in the appropriate framework of the weak formulation of boundary value problems [11,12], proved convergence to the linear plate model [22] in the context of linearized elasticity, and obtained formally the von Kármán plate model from finite elasticity [8]. See also [55,56] for the time-dependent case and [9] for a comprehensive list of references.

The issue of deriving two-dimensional models valid for large deformations, by means of an asymptotic formalism, was subsequently tackled by Fox, Simo and the second author in [26]. They showed, in the context of the Saint Venant–Kirchhoff materials subject to appropriate boundary conditions, how to recover a hierarchy of four models. This hierarchy, driven by the order of magnitude of the applied loads, consisted of: the nonlinear membrane model, the inextensional bending model, the von Kármán model and the linear plate model. The models thus obtained still required a justification through rigorous convergence results. In [35], Le Dret and the second author used the variational point of view and proved Γ-convergence of the 3-dimensional elastic energies to a nonlinear membrane energy, valid for loads of magnitude of order 1. We remark that the expression of the limiting stored energy therein consisted of quasiconvexification of the 3d energy, first minimized with respect to normal stretches. This allowed to recover the degeneracy under compression; a feature that is otherwise missed by formal expansions. We further mention that for 3d → 1d reduction, a similar point of view had been introduced by Acerbi, Buttazzo and Percivale in [1].

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