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Analysis of the loss of boundary conditions for the diffusive Hamilton–Jacobi equation

Alessio Porretta [a], Philippe Souplet [b,*]

[a] Università di Roma Tor Vergata, Dipartimento di Matematica, Via della Ricerca Scientifica 1, 00133 Roma, Italy
[b] Université Paris 13, Sorbonne Paris Cité, CNRS UMR 7539, Laboratoire Analyse Géométrie et Applications, 93430 Villetaneuse, France

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Abstract

We consider the diffusive Hamilton–Jacobi equation, with superquadratic Hamiltonian, homogeneous Dirichlet conditions and regular initial data. It is known from [4] (Barles–DaLio, 2004) that the problem admits a unique, continuous, global viscosity solution, which extends the classical solution in case gradient blowup occurs. We study the question of the possible loss of boundary conditions after gradient blowup, which seems to have remained an open problem till now.

Our results show that the issue strongly depends on the initial data and reveal a rather rich variety of phenomena. For any smooth bounded domain, we construct initial data such that the loss of boundary conditions occurs everywhere on the boundary, as well as initial data for which no loss of boundary conditions occurs in spite of gradient blowup. Actually, we show that the latter possibility is rather exceptional. More generally, we show that the set of the points where boundary conditions are lost, can be prescribed to be arbitrarily close to any given open subset of the boundary.

Keywords: Diffusive Hamilton–Jacobi equation; Viscosity solution; Gradient blow-up; Loss of boundary conditions

1. Introduction

We consider the initial-boundary value problem for the diffusive Hamilton–Jacobi equation:

$$\begin{cases} u_t - \Delta u = |\nabla u|^p, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.1}$$

Throughout this article, we assume that Ω is a $C^{2+\alpha}$ smooth bounded domain of $\mathbb{R}^n$, $p > 2$ and

$$u_0 \in X := \left\{ v \in C^1(\overline{\Omega});\ v \ge 0 \text{ and } v = 0 \text{ on } \partial\Omega \right\},$$

* Corresponding author.
E-mail addresses: porretta@mat.uniroma2.it (A. Porretta), souplet@math.univ-paris13.fr (Ph. Souplet).

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endowed with the C^1 norm. This problem has been studied by many authors in the past decades (see e.g. [22, Chapter 40] and the references therein).

By standard theory [10], it is known that problem (1.1) admits a unique, maximal **classical C^1 solution** $u \geq 0$, such that $u \in C^{2,1}(\overline{\Omega} \times (0, T^*))$ and $u, \nabla u \in C(\overline{\Omega} \times [0, T^*))$. Here $T^* = T^*(u_0) \in (0, \infty]$ denotes its existence time and the differential equation and the boundary conditions are satisfied in the pointwise sense for $t \in (0, T^*)$. Moreover, the solution satisfies the maximum principle estimate

$$\|u(t)\|_\infty \leq \|u_0\|_\infty, \quad 0 < t < T^*,$$

and the classical C^1 solution can only cease to exist through **gradient blowup**:

$$T^* < \infty \quad \Longrightarrow \quad \lim_{t \to T^*} \|\nabla u(t)\|_\infty = \infty.$$

Actually, ∇u remains bounded away from the boundary and gradient blowup occurs only on $\partial\Omega$ (see [25]). Furthermore it is known (see, e.g., [1,2,23]) that $T^* < \infty$ whenever the initial data is suitably large. We also recall that this phenomenon does not occur when $1 \leq p \leq 2$.

On the other hand, it was proved in [4] that problem (1.1) admits a unique **global viscosity solution** $u \in C(\overline{\Omega} \times [0, \infty))$, where the boundary conditions have to be understood in the viscosity sense. Throughout this article, we shall denote this solution by u without risk of confusion, since the two solutions coincide on $[0, T^*)$. (The result in [4] is actually valid for any $u_0 \in C_0(\overline{\Omega})$, but this need not concern us here.) Moreover, u is actually smooth away from the boundary, namely

$$u \in C^{2,1}(\Omega \times (0, \infty))$$

and it solves the PDE in (1.1) in the classical sense in $\Omega \times (0, \infty)$ (see Section 3 for details). It was next proved in [20] that for $t > T_0 = T_0(\|u_0\|_\infty)$ sufficiently large, u is actually a classical solution again, namely $u \in C^{2,1}(\overline{\Omega} \times (T_0, \infty))$ with $u(\cdot, t) = 0$ on $\partial\Omega$.

When gradient blowup occurs, the question of possible loss of boundary conditions for $t \geq T^*$ (hence actually in $[T^*, T_0]$) has remained essentially open. Namely, it is unknown whether or not u satisfies the boundary conditions $u = 0$ on $\partial\Omega \times [T^*, T_0]$ in the classical sense. In what follows, we say that **loss of boundary conditions** occurs at a point $x_0 \in \partial\Omega$ if $u(x_0, t) > 0$ for some $t \geq T^*$.

The goal of this article is to give some answers to this question. A main conclusion is that loss of boundary conditions after gradient blowup **may or may not occur, depending on the initial data.** Furthermore, in case it occurs, the structure and size of the set of the points where boundary conditions are lost, strongly depends on the initial data. This is somewhat surprising and shows that the problem reveals a rather rich variety of phenomena.

Throughout this paper, we denote by φ_1 the first Dirichlet eigenfunction of $-\Delta$ in Ω, normalized by $\int_\Omega \varphi_1 \, dx = 1$.

2. Main results

For any $u_0 \in X$, we define the **loss of boundary conditions set** by

$$\mathcal{L}(u_0) = \left\{ x_0 \in \partial\Omega,\ u(x_0, t) > 0 \text{ for some } t > 0 \right\}.$$

Our first result shows that there exist initial data for which the loss of boundary conditions occurs **everywhere on** $\partial\Omega$, and moreover can be achieved at a common time.

Theorem 1. *Let $p > 2$. There exists $u_0 \in X$ such that $\mathcal{L}(u_0) = \partial\Omega$ and that, moreover,*

$$u(x, t_0) \geq c_0, \quad x \in \partial\Omega,$$

for some $t_0, c_0 > 0$. Furthermore, the same remains true for any $v_0 \in X$ with $v_0 \geq u_0$.

Our next result shows that, at the opposite, there are gradient blowup solutions for which **no loss of boundary conditions ever occurs.**

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