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# Stable blowup for wave equations in odd space dimensions ☆

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**Abstract**

We consider semilinear wave equations with focusing power nonlinearities in odd space dimensions $d \geq 5$. We prove that for every $p > \frac{d+3}{d-1}$ there exists an open set of radial initial data in $H^{\frac{d+1}{2}} \times H^{\frac{d-1}{2}}$ such that the corresponding solution exists in a backward lightcone and approaches the ODE blowup profile. The result covers the entire range of energy supercritical nonlinearities and extends our previous work for the three-dimensional radial wave equation to higher space dimensions.


*Keywords:* Nonlinear wave equations; Blowup; Stability

## 1. Introduction

We consider the initial value problem for the focusing nonlinear wave equation

$$\begin{aligned} &\partial_t^2 u - \Delta u = |u|^{p-1} u, \\ &u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1, \end{aligned} \tag{1.1}$$

for $(t, x) \in I \times \mathbb{R}^d$, $d = 2k + 1$, $k \geq 2$ and $I$ an interval, where $0 \in I$. Eq. (1.1) is conformally invariant for $p = \frac{d+3}{d-1}$ and we restrict ourselves to the *superconformal* case

$$p > \frac{d+3}{d-1}. \tag{1.2}$$

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The above equation enjoys scaling invariance in the sense that if $u$ solves Eq. (1.1) then another solution can be obtained by setting $u_\lambda(t,x) := \lambda^{-\frac{2}{p-1}} u(t/\lambda, x/\lambda)$ for $\lambda > 0$. The conserved energy is given by

$$E(u(t,\cdot), \partial_t u(t,\cdot)) = \tfrac{1}{2}\|(u(t,\cdot), \partial_t u(t,\cdot))\|^2_{\dot{H}^1 \times L^2(\mathbb{R}^d)} - \tfrac{1}{p+1}\|u(t,\cdot)\|^{p+1}_{L^{p+1}(\mathbb{R}^d)}$$

and it is invariant under the above scaling for $p = \frac{d+2}{d-2}$, which defines the *energy critical* case. In general, the scaling invariant Sobolev spaces are $\dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^d)$, where the index $s_p = \frac{d}{2} - \frac{2}{p-1}$ is usually referred to as the critical regularity.

### 1.1. Basic well-posedness theory and explicit blowup solutions

One is usually interested in (strong) solutions of Eq. (1.1) that satisfy the equation in integral form by using Duhamel's principle, see for example [45]. In this sense, Eq. (1.1) is locally well-posed in $\dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^d)$ for $d \geq 5$ and $p > \frac{d+3}{d-1}$, given that the nonlinearity is sufficiently regular, cf. Lindblad and Sogge [32]. Moreover, solutions that correspond to sufficiently small initial data can be extended globally in time. We also note that local well-posedness in $H^s \times H^{s-1}(\mathbb{R}^d)$ for $s > \frac{d}{2}$ and smooth nonlinearities is classical [45]. However, global well-posedness does not hold in general. A convexity argument by Levine [31] shows that initial data with negative energy (and finite $L^2$-norm) lead to blowup in finite time, cf. also [24] for generalizations.

Explicit examples for singularity formation can be obtained by considering the so called ODE blowup solution

$$u_T(t,x) = c_p (T-t)^{-\frac{2}{p-1}}, \quad c_p := \left[\frac{2(p+1)}{(p-1)^2}\right]^{\frac{1}{p-1}}, \tag{1.3}$$

which is independent of the space dimension and solves the ordinary differential equation $u_{tt} = |u|^{p-1}u$ for $p > 1$. By finite speed of propagation one can use $u_T$ to construct compactly supported smooth initial data such that the solution blows up as $t \to T$.

In one space dimension the ODE blowup mechanism is universal, cf. the fundamental work by Merle and Zaag [36,37,40,39] and the references therein. In higher dimensions, the situation is more complex. Depending on $d$ and $p$ many other explicit examples for singular solutions were found in the past years, including the celebrated work of Krieger, Schlag and Tataru [29] on *type II* blowup solutions for the energy critical equation in three space dimensions, see below. For $d = 3$, $p = 3$ and $p \geq 7$ an odd integer, it was proved by Bizoń, Breitenlohner, Maison and Wasserman [4,3] that Eq. (1.1) admits infinitely many radial *self-similar* blowup solutions of the form $(T-t)^{-\frac{2}{p-1}} f_n(\frac{|x|}{T-t})$, $n \in \mathbb{N}_0$, with $u_T$ corresponding to the groundstate, i.e., $f_0 = c_p$. Another blowup mechanism for Eq. (1.1), which only exists for $d \geq 11$ and a range of supercritical nonlinearities $p > p(d) > \frac{d+2}{d-2}$, was recently established by Collot [6], see below.

Most of these explicit solutions have unstable directions, i.e., they are unstable under generic small perturbations and are not supposed to describe the 'typical' blowup behavior for solutions of Eq. (1.1), see for example [26]. On the other hand, numerical experiments by Bizoń, Chmaj and Tabor [2] for the three-dimensional equation show that the behavior of generic radial blowup solutions can be characterized in terms of the ODE blowup solution locally around the blowup point.

The stability of $u_T$ in three space dimensions was established in our previous works [10,12] for radial perturbations and all $p > 1$. Recently, we could extend this to the general case (without symmetry) [11] for $p > 3$. For subconformal nonlinearities the dynamics around $u_T$ were also investigated by Merle and Zaag [41] in the non-radial setting and in arbitrary space dimensions.

In view of the findings in [6] for supercritical radial wave equations in high space dimensions, we extend our previous results and establish the stability of the ODE blowup solution in arbitrary odd space dimensions. Although for $d = 3$ we were able to drop the symmetry assumption, it is an open question how this can be accomplished for $d \geq 5$, see the discussion below. We therefore restrict ourselves to the radial case and study solutions that blow up at the origin (which is the most interesting case).

We note that this work is not a mere technical generalization of [10,12]. It can rather be viewed as a systematization and refinement of our approach that has also been applied (with slight modifications) to establish stable self-similar blowup for equivariant wave maps [13,10] and Yang–Mills fields [8] in supercritical dimensions.

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