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A characterization result for the existence of a two-phase material minimizing the first eigenvalue

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Abstract

Given two isotropic homogeneous materials represented by two constants $0<\alpha<\beta$ in a smooth bounded open set $\Omega\subset\mathbb{R}^N$, and a positive number $\kappa<|\Omega|$, we consider here the problem consisting in finding a mixture of these materials $\alpha\chi_\omega+\beta(1-\chi_\omega)$, $\omega\subset\mathbb{R}^N$ measurable, with $|\omega|\leq\kappa$, such that the first eigenvalue of the operator $u\in H^1_0(\Omega)\to-\text{div}\big((\alpha\chi_\omega+\beta(1-\chi_\omega))\nabla u\big)$ reaches the minimum value. In a recent paper, [6], we have proved that this problem has not solution in general. On the other hand, it was proved in [1] that it has solution if Ω is a ball. Here, we show the following reciprocate result: If $\Omega\subset\mathbb{R}^N$ is smooth, simply connected and has connected boundary, then the problem has a solution if and only if Ω is a ball.

MSC: 49J20

Keywords: Two-phase material; Control in the coefficients; Eigenvalue; Non-existence

1. Introduction

We consider a bounded open set $\Omega\subset\mathbb{R}^N$, $N\geq 2$, and two constants $0<\alpha<\beta$, representing two homogeneous isotropic materials (thermic, electric, elastic,...). A classical problem in optimal design consists in mixing these materials in order to minimize a certain functional. Such as it is proved in [17] and [18], this type of problems has not solution in general and then it is usual to deal with relaxed formulations which can be obtained by using the homogenization theory (see e.g. [2,7,19,22,23]).

Between the most studied problems of this type (see e.g. [2,6,14,15,19]) we emphasize the following one

$$\begin{cases}\min\displaystyle\int_\Omega(\alpha\chi_\omega+\beta\chi_{\Omega\setminus\omega})|\nabla u|^2dx\\ -\text{div}\big((\alpha\chi_\omega+\beta\chi_{\Omega\setminus\omega})\nabla u\big)=f\ \text{ in }\Omega,\quad u=0\ \text{ on }\partial\Omega,\quad|\omega|<\kappa,\end{cases}\tag{1.1}$$

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with $f \in H^{-1}(\Omega)$ and $\kappa \in (0, |\Omega|)$ (for $\kappa \geq |\Omega|$, the solution is the trivial one $\omega = \Omega$). A special attention has been paid for $f = 1$ and $N = 2$, where it represents the optimal distribution of two materials in the cross section a beam in order to minimize the torsion. In this case, it has been proved in [19] that if Ω is simply connected and smooth and there exists a smooth solution ω, then Ω is a ball. This result has been improved in [6] by showing that the result holds true without any smoothness assumptions on ω (the case $N > 2$ is also considered). The proof is based on certain regularity results for the solution of the relaxed formulation of (1.1) also obtained in [6]. A related problem consisting in replacing the minimum in (1.1) by a maximum has been considered in [5].

It has also been observed in [6] that problem (1.1) is strongly related to another classical optimization design problem for a two-phase material. It consists in finding a measurable set $\omega \subset \Omega$ with $|\omega| \leq \kappa$ ($0 < \kappa < |\Omega|$ as above) such that the first eigenvalue of the operator

$$u \in H_0^1(\Omega) \mapsto -\text{div}\left((\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega})\nabla u\right) \in H^{-1}(\Omega) \tag{1.2}$$

becomes minimal. Namely, it is proved that the relaxed formulation of this problem is equivalent to solve the relaxed formulation of (1.1) for every $f \in L^2(\Omega)$ with $\|f\|_{L^2(\Omega)} = 1$ and then to minimize in f. Thus, the regularity results proved in [6] for (1.1) also hold for the minimization of the eigenvalue. As an application, it has been shown that the problem has not solution if Ω is a rectangle or an ellipsis. On the other hand, it was proved in [1] that the eigenvalue problem has a solution in the particular case where Ω is a ball and then the optimal set ω has a radial structure. Some discussions about the exact structure of ω when Ω is a ball can be found in [8,9,16] and [20].

The purpose of the present paper is to show that, similarly to the result stated above for problem (1.1) with $f = 1$, if Ω is a smooth simply connected open set with connected boundary such that the minimization of the first eigenvalue of the operator (1.2) has a solution, then Ω is a ball. As for problem (1.1), the proof uses the results obtained in [6] but the reasoning is more involved. For problem (1.1) with $f = 1$ one has that the optimal solutions (ω, u) are such that there exist an analytic function w and a positive number μ, satisfying

$$(\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega})\nabla u = \nabla w, \qquad \{|\nabla w| > \mu\} \subset \omega \subset \{|\nabla w| \geq \mu\}. \tag{1.3}$$

Moreover, the Laplacian of $|\nabla w|^2$ in Ω is nonnegative. For the eigenvalue problem, statement (1.3) still holds true but now w is non-analytic and the Laplacian of $|\nabla w|^2$ can change its sign in Ω. Thus, many of the ideas used in [6] (and [19]) cannot be used here.

2. The characterization result

For a smooth bounded open set $\Omega \subset \mathbb{R}^N$, $N \geq 2$, and three constants $0 < \alpha < \beta$, $0 < \kappa < |\Omega|$, we consider the problem consisting in finding a measurable subset ω of Ω with $|\omega| \leq \kappa$, such that the first eigenvalue of the problem

$$\begin{cases} -\text{div}\left((\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega})\nabla u\right) = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

reaches its minimum value. This can also been formulated as

$$\begin{cases} \min \displaystyle\int_\Omega (\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega})|\nabla u|^2 dx \\ u \in H_0^1(\Omega), \quad \displaystyle\int_\Omega |u|^2 dx = 1 \\ \omega \subset \Omega \text{ measurable}, \quad |\omega| \leq \kappa. \end{cases} \tag{2.2}$$

We remark that if we do not assume the volume restriction $|\omega| \leq \kappa$, then the solution is the trivial one given by $\omega = \Omega$. However in the applications, the material α can be more expensive than β and thus, we can only dispose of a certain quantity κ of material α. The question then is how to distribute it in an optimal way.

As an application of (2.2) we can consider the following problem for the heat equation

$$\begin{cases} \partial_t u - \text{div}\left((\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega})\nabla u\right) = 0 & \text{in } \Omega \times (0, \infty) \\ u = 0 & \text{on } \partial\Omega \times (0, \infty) \\ u_{|t=0} = u_0, \end{cases}$$

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