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ANIHPC:2810

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Small amplitude periodic solutions of Klein–Gordon equations

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Received 21 January 2015; received in revised form 4 October 2016; accepted 13 October 2016

Abstract

We consider a class of nonlinear Klein–Gordon equations $u_{tt} = u_{xx} - u + f(u)$ and obtain a family of small amplitude periodic solutions, where the temporal and spatial period have different scales. The proof is based on a combination of Lyapunov–Schmidt reduction, averaging and Nash–Moser iteration.

Keywords: Klein–Gordon equation; Periodic solution; Nash–Moser iteration

1. Introduction

The nonlinear Klein–Gordon equation

$$u_{tt} = u_{xx} - u + f(u)\ , \ x \in \mathbb{R}, \tag{1.1}$$

is an important model in particle physics, which models the field equation for spineless particles. Classical examples include Sine-Gordon equation and ϕ^4-model. The main result of this paper is to construct a family of small amplitude periodic (both in time and space) solutions of (1.1), where the temporal and spatial period have different scales. Moreover, we can approximate such periodic solutions by a simple periodic orbit for a planar system up to exponentially small errors. We will postpone the precise statement until the end of Section 2 after we introduce some mathematical notations. Throughout this paper, we will assume the nonlinear term f to be analytic and odd in u. The analyticity is crucial for us to prove the exponentially small error. The oddness is assumed just for convenience. We will comment on how to deal with general f containing quadratic terms later in this section.

The motivation of this paper originates from the sine-Gordon equation ($f(u) = u - \sin u$)

$$u_{tt} = u_{xx} - \sin u, \tag{1.2}$$

which has a family of time periodic solutions (breathers)

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[1] The author is funded in part by NSF contracts DMS-1500159 and DMS-1645673.

http://dx.doi.org/10.1016/j.anihpc.2016.10.002
0294-1449/

Please cite this article in press as: N. Lu, Small amplitude periodic solutions of Klein–Gordon equations, Ann. I. H. Poincaré – AN (2016), http://dx.doi.org/10.1016/j.anihpc.2016.10.002

$$u(x,t) = 4\arctan\frac{\sqrt{1-\omega^2}\sin\omega t}{\omega\cosh\sqrt{1-\omega^2}x}. \tag{1.3}$$

Clearly, the above formula is only defined for $|\omega| < 1$. Since (1.1) can be viewed as a perturbation of (1.2) for small amplitude solutions, it is natural to ask if (1.1) admits any time periodic solution parameterized by ω. The author studied the problem for $\omega = \sqrt{1-\epsilon^2}$ in [19], where he obtained small amplitude (of order ϵ) breather solutions with $O(e^{-\frac{c}{\epsilon}})$ tails as $|x| \to \infty$, i.e., the solution is $\frac{2\pi}{\omega}$ periodic in time and almost localized in space with exponentially small errors. In this manuscript, we continue our study for $\omega = \sqrt{1+\epsilon^2}$. It turns out the solutions we obtain here have completely different behavior in spatial variable, namely, the solution is also periodic in x.

Since the temporal period is explicitly known, we use the spatial dynamics method (interchanging x and t) to rewrite (1.1) as a nonlinear wave equation with periodic boundary condition

$$u_{tt} = u_{xx} + u - f(u)\ ,\ u(x,t) = u(x+\frac{2\pi}{\omega}, t), \tag{1.4}$$

where $\omega = \sqrt{1+\epsilon^2}$. By normalizing the spatial period (temporal period for (1.1)) to be 2π, i.e. rescale x to ωx, we further transform (1.4) to

$$u_{tt} = \omega^2 u_{xx} + u - f(u)\ ,\ u(x,t) = u(x+2\pi, t). \tag{1.5}$$

Since the nonlinearity f is odd in u, it suffices to restrict u to be odd in x. Consequently, the linear operator $\omega^2\partial_{xx} + 1$ has characteristic frequencies $\pm\epsilon i$ and $\pm\sqrt{\omega^2k^2-1}i$ for $k \geq 2$ with multiplicity 1.

The strategy for finding periodic solutions of (1.1) ((1.5) under spatial dynamics formulation) is a combination of singular perturbation theory, averaging, Lyapunov–Schmidt reduction and Nash–Moser iteration. First of all, we observe that the characteristic frequencies of the linear operator $\omega^2\partial_{xx} + 1$ have two scales, namely, one pair of $O(\epsilon)$-eigenvalues and infinitely many pairs of $O(1)$-eigenvalues. To obtain uniform knowledge in ϵ, we rescale time in (1.5) to blow up small eigenvalues from $O(\epsilon)$ to $O(1)$, which makes the $O(1)$-eigenvalues become $O(\frac{1}{\epsilon})$. With appropriate spatial rescaling, we obtain a singularly perturbed system (2.3) and (2.4). The singular limit of such system can be rigorously justified as a second order ordinary differential equation (2.11) whose phase plane contains a lot of periodic orbits. Secondly, we perform a sequence of partial normal form transformations to obtain a system whose solutions are exponentially close to the limit equation. Finally, we follow the Lyapunov–Schmidt type argument in [28] and the Nash–Moser iteration in [6] to find our periodic solutions near those unperturbed ones.

The problem of finding periodic solutions to Hamiltonian PDEs has been extensively studied since the 1960s, see for example [5,6,16,17,24–27] and references therein. The first breakthrough on this problem was due to Rabinowitz [24]. He rephrased the problem as a variational problem and proved the existence of periodic solutions under the monotonicity assumption on the nonlinearity whenever the time period was a rational multiple of the length of the spatial interval. Subsequently, many authors, such as Brézis, Coron, Nirenberg etc., have used and developed Rabinowitz's variational methods to obtain related results, see [2,8,10]. In these papers, the time period T is required to be a rational multiple of π. The case in which T is some irrational multiple of π has been investigated by Fečkan [13] and McKenna [20]. At the end of the 1980s, a different approach which used the Kolmogorov–Arnold–Moser (KAM) theory was developed from the viewpoint of infinite dimensional dynamical systems by Kuksin [18] and Wayne [29]. This method allows one to obtain solutions whose periods are irrational multiples of the length of the spatial interval, and it can also be easily extended to construct quasi-periodic solutions see [23,15,30] and references therein. Unlike the variational techniques, the KAM theory only yields solutions of small amplitude. Later, in the original work of Craig–Wayne [12], the existence of periodic solutions for the one-dimensional conservative nonlinear wave equation was also proved by using the Lyapunov–Schmidt method and Newton iterations. Here we point out equation (1.1) is not completely resonant. Results on periodic/quasi-periodic solutions for completely resonant nonlinear wave equations can be found in [1,4,14]. For exponential stability of periodic solutions, we refer readers to Bambusi–Nekhoroshev [9] and Paleari–Bambusi–Cacciatori [22] and references therein.

The methodology employed in this paper is based on a perturbation argument, which is different from the variational technique and the KAM theory. Even though our solutions still have small amplitudes, which is due to scaling, we actually obtain them from some unperturbed periodic orbits which have large amplitudes. The central idea of KAM theory is to use successive approximate solutions (obtained by normal form transformations) with better accuracy to obtain the exact solution. This method usually requires the analyticity of nonlinearity to assure the convergence of

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