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Geometry of minimizers for the interaction energy with mildly repulsive potentials

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Abstract

We show that the support of any local minimizer of the interaction energy consists of isolated points whenever the interaction potential is of class C^2 and mildly repulsive at the origin; moreover, if the minimizer is global, then its support is finite. In addition, for some class of potentials we prove the validity of a uniform upper bound on the cardinal of the support of a global minimizer. Finally, in the one-dimensional case, we give quantitative bounds.

Keywords: Interaction energy; Local minimizers; Mild repulsion

1. Introduction

Consider the *interaction energy* $E : \mathcal{P}(\mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}$, defined on the set of Borel probability measures $\mathcal{P}(\mathbb{R}^n)$ by

$$E(\mu) = \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} W(x-y)\, d\mu(x)\, d\mu(y) \quad \text{for all } \mu \in \mathcal{P}(\mathbb{R}^n), \tag{1.1}$$

where $W : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is an *interaction potential*. In recent years, the study of local and global minimizers of energies of the form (1.1) has been in the spotlight of applied mathematics, in particular in the context of the variational approach to partial differential equations. The main reason for this interest is that E is a Lyapunov functional for the continuity equation

$$\partial_t \mu = \nabla \cdot (\mu \nabla W * \mu) \quad \text{on } \mathbb{R}^n, \text{ for } t > 0,$$

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called the *aggregation equation*, where $*$ is the convolution operator and $\mu\colon [0,\infty) \to \mathcal{P}(\mathbb{R}^n)$ is here a probability curve. These equations describe the continuum behavior of agents interacting via the potential W, and are at the core of many applications ranging from mathematical biology to granular media and economics, see [16,14,11,17,4] and the references therein. They can also be obtained as dissipative limits of hydrodynamic equations for collective behavior [13].

Typically, interaction potentials are repulsive towards the origin and attractive towards infinity; this reproduces the "social", or natural, behavior of the agents that are usually considered in applications. In [2] the authors showed that the dimension of the support of a minimizer of E is directly related to the repulsiveness of the potential at the origin, i.e., to the strength of the repulsion of two very close particles. More precisely, the stronger the repulsion (up to Newtonian), the higher the dimension of the support. In particular, in the case of *mild repulsion*—when the potential behaves like a power of order α, with $\alpha > 2$, near the origin—the Hausdorff dimension of each smooth enough component of the support has to be zero, see [2, Theorem 2]. The smoothness assumption on the connected components of the support was essential in the proof, hence several open problems immediately arise: Is it possible to have minimizers whose supports lie on sets of non-integer Hausdorff dimension? Or can the support have integer dimension but non-smooth components? And, assuming one can prove that it is of zero dimension, is the support discrete?

In this work we give a conclusive answer to all these questions. Let us first mention that extensive simulations [10,12,3,2,1,8] showed that fractal supports were not numerically observed and, moreover, these numerical simulations were consistently giving minimizers supported on finite numbers of points. This is precisely the rigorous result we show in this work under suitable assumptions on the repulsive-attractive potential W. In the rest of the paper we always assume that W is radially symmetric, of class C^2, and such that

$$W(0) = 0, \text{ and there is } R > 0 \text{ s.t. } W(x) < 0 \text{ if } 0 < |x| < R, \text{ and } W(x) \geq 0 \text{ if } |x| \geq R. \tag{1.2}$$

Let us write $w(|x|) := W(x)$ for all $x \in \mathbb{R}^n$. Remark that, by convention, w being repulsive (resp. attractive) at $r > 0$ means $w'(r) < 0$ (resp. $w'(r) > 0$). We suppose that W is mildly repulsive, that is,

$$\text{there exist } \alpha \geq 2 \text{ and } C > 0 \text{ such that } w'(r)r^{1-\alpha} \to -C \text{ as } r \to 0. \tag{1.3}$$

Note that, since $w(0) = 0$, (1.3) implies that $w(r)r^{-\alpha} \to -C/\alpha$ as $r \to 0$.

When we refer to minimizers of the energy (1.1), we either refer to *global* minimizers, in which case no underlying topology is required, or to *local* minimizers, in which case we need to specify the topology. In [2] it was proven that a natural topology to obtain suitable Euler–Lagrange conditions is that induced by the ∞-*Wasserstein distance*. We define the ∞-Wasserstein distance between two probability measures μ and ν by

$$d_\infty(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \sup_{(x,y)\in \operatorname{supp}\pi} |x - y|, \tag{1.4}$$

where $\Pi(\mu, \nu)$ is the space of probability measures on $\mathbb{R}^n \times \mathbb{R}^n$ with first marginal μ and second marginal ν. We therefore say that $\mu \in \mathcal{P}(\mathbb{R}^n)$ is a d_∞*-local minimizer* of E if there exists $\varepsilon > 0$ such that $E(\mu) \leq E(\nu)$ for all $\nu \in \mathcal{P}(\mathbb{R}^n)$ with $d_\infty(\mu, \nu) < \varepsilon$. We refer to [2,7,18] and the references therein for a good account on the properties of this distance and its relation to more classical metrics in optimal transport.

The Euler–Lagrange conditions for d_∞-local minimizers of E were used to give necessary and sufficient conditions on repulsive-attractive potentials to have existence of global minimizers [5,15], and to analyze the regularity of the d_∞-local minimizers for potentials which are as repulsive as, or more singular than, the Newtonian potential [7]. In both cases, d_∞-local minimizers are solutions of some related obstacle problems for Laplacian or nonlocal Laplacian operators, implying that they are bounded and smooth in their supports, or even continuous up to the boundary [7,9]. Similar Euler–Lagrange equations were also used for nonlinear versions of the Keller–Segel model in order to characterize minimizers of related functionals [6].

Our main theorem in this work is the following.

Theorem 1.1. *Let $\mu \in \mathcal{P}(\mathbb{R}^n)$ be a d_∞-local minimizer of the interaction energy with $E(\mu) < \infty$. Then every point in the support of μ is isolated. If moreover μ is a global minimizer, then the support of μ consists of finitely many points.*

In Section 2 we show some preliminary results concerning minimizers that are needed for the proof of Theorem 1.1, which we give in Section 3. In Section 4 we prove upper estimates on the cardinal of the support of a global minimizer.