ARTICLE IN PRESS
ANIHPC:2813

Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN ••• (••••) •••–•••

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Fujita blow up phenomena and hair trigger effect: The role of dispersal tails

Matthieu Alfaro

IMAG, Université de Montpellier, CC051, Place Eugène Bataillon, 34095 Montpellier Cedex 5, France

Received 9 May 2016; received in revised form 4 October 2016; accepted 13 October 2016

Abstract

We consider the nonlocal diffusion equation $\partial_t u = J * u - u + u^{1+p}$ in the whole of $\mathbb{R}^N$. We prove that the Fujita exponent dramatically depends on the behavior of the Fourier transform of the kernel J near the origin, which is linked to the tails of J. In particular, for compactly supported or exponentially bounded kernels, the Fujita exponent is the same as that of the nonlinear Heat equation $\partial_t u = \Delta u + u^{1+p}$. On the other hand, for kernels with algebraic tails, the Fujita exponent is either of the Heat type or of some related Fractional type, depending on the finiteness of the second moment of J. As an application of the result in population dynamics models, we discuss the hair trigger effect for $\partial_t u = J * u - u + u^{1+p}(1-u)$.

MSC: 35B40; 35B33; 45K05; 47G20

Keywords: Blow up solution; Global solution; Fujita exponent; Nonlocal diffusion; Dispersal tails; Hair trigger effect

1. Introduction

In this work we consider solutions $u(t,x)$ to the nonlinear ($p>0$) partial integro-differential equation

$$\partial_t u = J * u - u + u^{1+p} \quad \text{in } (0,\infty)\times\mathbb{R}^N, \tag{1}$$

in any dimension $N \geq 1$. Equation (1) is supplemented with a nonnegative and nontrivial initial data, and we aim at determining the so-called *Fujita exponent* p_F, that is the value of p that separates "systematic blow up solutions" from "blow up solutions vs global and extincting solutions" (see below for details). We shall prove that the Fujita exponent dramatically depends on the behavior of the Fourier transform of the kernel J near the origin, which is linked to the tails of J. Depending on these tails, it turns out that the Fujita phenomenon in (1) can be similar to that of the nonlinear Heat equation, or to that of a related nonlinear Fractional equation.

As an application of our main result, we consider

$$\partial_t u = J * u - u + u^{1+p}(1-u) \quad \text{in } (0,\infty)\times\mathbb{R}^N, \tag{2}$$

E-mail address: matthieu.alfaro@umontpellier.fr.

http://dx.doi.org/10.1016/j.anihpc.2016.10.005
0294-1449/

which serves as a population dynamics model where both long range dispersal (via the kernel J) and a weak Allee effect (via the degeneracy of the steady state $u \equiv 0$, due to $p > 0$) are taken into account. Depending on the balance between the tails of J and the strength of the Allee effect, we discuss the so-called *hair trigger effect* — meaning that any small perturbation from $u \equiv 0$ drives the solution to $u \equiv 1$ — or the possibility of extincting solutions.

In his seminal work [10], Fujita considered solutions $u(t,x)$ to the nonlinear Heat equation

$$\partial_t u = \Delta u + u^{1+p} \quad \text{in } (0,\infty) \times \mathbb{R}^N, \tag{3}$$

supplemented with a nonnegative and nontrivial initial data. For such a problem, the Fujita exponent is $p_F = \frac{2}{N}$. Precisely, if $0 < p \leq p_F$ then any solution blows up in finite time; if $p > p_F$ then solutions with large initial data blow up in finite time whereas solutions with small initial data are global in time and go extinct as $t \to \infty$. For a precise statement we refer to [10] for the cases $0 < p < p_F$ and $p > p_F$. The critical case $p = p_F$ is studied in [15] when $N = 1, 2$, in [17] when $N \geq 3$, and in [24] via a direct and simpler approach. Let us observe that, as is well known, solutions to the Heat equation $\partial_t u = \Delta u$ tend to zero as $t \to \infty$ like $\mathcal{O}\left(t^{-\frac{N}{2}}\right)$, which is a formal argument to guess $p_F = \frac{2}{N}$.

Since then, the Fujita phenomenon has attracted much interest and the literature on refinements of the results or on various local variants of equation (3) is rather large. Let us mention for instance the works [24,19,7,21], or [22] for an overview, and the references therein.

When the Laplacian diffusion operator is replaced by the Fractional Laplacian, the situation is also well understood: the Fujita exponent for

$$\partial_t u = -(-\Delta)^{s/2} u + u^{1+p} \quad \text{in } (0,\infty) \times \mathbb{R}^N, \quad 0 < s \leq 2, \tag{4}$$

is $p_F = \frac{s}{N}$. We refer to the work of Sugitani [23]. See also, among others, [5] for a probabilistic approach, and [13] for a variant of (4). Let us observe that, as is well known, solutions to the Fractional diffusion equation $\partial_t u = -(-\Delta)^{s/2} u$ tend to zero as $t \to \infty$ like $\mathcal{O}\left(t^{-\frac{N}{s}}\right)$, which is again a formal argument to guess $p_F = \frac{s}{N}$.

As far as we know, much less is known for the nonlocal equation (1). Let us mention the work of García-Melián and Quirós [11] (and [26] for a variant) who treat the case of compactly supported dispersal kernel J. In such a situation, the Fujita exponent for (1) is the same as that of (3), namely $p_F = \frac{2}{N}$. In order to take into account rare long-distance dispersal events, which are relevant in many population dynamics models (seeds dispersal for instance), we allow in this work kernels J which have nontrivial tails. Two typical situations are when J has (light) exponential tails or (heavy) algebraic tails, the latter case meaning

$$J(x) \sim \frac{C}{|x|^\alpha} \quad \text{as } |x| \to \infty, \quad \text{with } \alpha > N. \tag{5}$$

Owing to the decay of solutions to $\partial_t u = J * u - u$ proved by Chasseigne, Chaves and Rossi [6], we guess that $p_F = \frac{2}{N}$ in the exponential case, whereas

$$p_F = \begin{cases} \frac{\alpha}{N} - 1 & \text{if } N < \alpha \leq N+2 \\ \frac{2}{N} & \text{if } \alpha > N+2, \end{cases} \tag{6}$$

in the algebraic case (5). In other words, in the algebraic case $\alpha > N+2$ the Fujita exponent is of the Heat type (3) (and so in the exponential case), but in the algebraic case $N < \alpha \leq N+2$ the Fujita exponent becomes of the Fractional type (4) with $s = \alpha - N \in (0, 2]$. This is the role of the present paper to prove, among others, these results.

Let us comment on some technical difficulties arising from (1). Notice first that, as far as (3) and (4) are concerned, some self-similarity properties of both the Heat kernel and the fundamental solution associated to the Fractional Laplacian may be quite helpful, as seen in [23] or [24]. Those self-similarity properties are not shared by the fundamental solution of $\partial_t u = J * u - u$. Secondly, notice that, when J is compactly supported, the underlying nonlocal eigenvalue problem to (1) is rather well understood [12] and the authors in [11] took advantage of its rescaling properties. As far as we know, such informations are not available for more general dispersal kernels, as those we consider. We therefore have to adapt some techniques, in particular when dealing with blow up phenomena.

Please cite this article in press as: M. Alfaro, Fujita blow up phenomena and hair trigger effect: The role of dispersal tails, Ann. I. H. Poincaré – AN (2016), http://dx.doi.org/10.1016/j.anihpc.2016.10.005