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On condensate of solutions for the Chern–Simons–Higgs equation

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Abstract

This is the first part of our comprehensive study on the structure of doubly periodic solutions for the Chern–Simons–Higgs equation with a small coupling constant. We first classify the bubbling type of the blow-up point according to the limit equations. Assuming that all the blow-up points are away from the vortex points, we prove the non-coexistence of different bubbling types in a sequence of bubbling solutions. Secondly, for the CS type bubbling solutions, we obtain an existence result without the condition on the blow-up set as in [4]. This seems to be the first general existence result of the multi-bubbling CS type solutions which is obtained under nearly necessary conditions. Necessary and sufficient conditions are also discussed for the existence of bubbling solutions blowing up at vortex points.

Keywords: Condensate; Chern–Simons–Higgs equation; Bubbling phenomena; Pohozaev identity; Uniqueness

1. Introduction

In the last decade, various Chern–Simons theories have been studied for their applications in different physics models, such as the relativistic Chern–Simons theory of superconductivity [11], Lozano–Marqueés–Moreno–Schaposnik model of bosonic sector of $N = 2$ super-symmetric Chern–Simons–Higgs theory [29], and Gudnason model of $N = 2$ super-symmetric Yang–Mills–Chern–Simons–Higgs theory [12], just to name a few. Those Chern–Simons systems, after a suitable ansatz, can be reduced to systems of elliptic partial differential equations with exponential nonlinearities. Although these nonlinear differential equations pose many analytically challenging problems and attract lots of attentions, there are still many problems unsolved. For the recently mathematical developments, we refer the readers to [1,2,5–8,13–16,18,22,23,27,28,30,33,32,37] and the references therein.

Among those non-trivial equations, the simplest one is the Abelian Chern–Simons–Higgs model proposed by Jackiw–Weinberg [19] and Hong–Kim–Pac [17]. The Chern–Simons–Higgs Lagrangian density is given by

$$\mathcal{L} = \frac{\kappa}{4}\epsilon^{\mu\nu\rho} F_{\mu\nu} A_\rho + D_\mu \phi \overline{D^\mu \phi} - \frac{1}{\kappa^2}|\phi|^2(1 - |\phi|^2)^2,$$

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where A_μ, $\mu=0,1,2$, is the gauge field in $\mathbb{R}^3$, $F_{\mu\nu}=\frac{\partial}{\partial^\mu}A_\nu-\frac{\partial}{\partial^\nu}A_\mu$ is the curvature tensor, ϕ is the Higgs field in $\mathbb{R}^3$, $D_\mu=\frac{\partial}{\partial^\mu}-\mathrm{i}A_\mu$, $\mathrm{i}=\sqrt{-1}$, is the gauge covariant derivative associated with A_μ, $\epsilon_{\mu\nu\rho}$ is the skew symmetric tensor with $\epsilon_{012}=0$ and the constant κ is the coupling constant. When the energy for the pair (ϕ, A) is saturated, in [19] and [17], the authors independently derived the following Bogomol'nyi type equations

$$(D_1+\mathrm{i}D_2)\phi=0, \tag{1.1}$$

and

$$F_{12}+\frac{2}{\kappa^2}|\phi|^2(1-|\phi|^2)^2=0. \tag{1.2}$$

Following Jaffe and Taubes [20], we can reduce (1.1) and (1.2) to a single elliptic equation as follows. Let $p_1,\cdots,p_N$ be a set of points in $\mathbb{R}^2$. We introduce a real valued function u and θ by $\phi=e^{\frac{1}{2}(u+\mathrm{i}\theta)}$ and $\theta=2\sum_{j=1}^N \arg(z-p_j)$, $z=x_1+\mathrm{i}x_2\in\mathbb{C}$. Then u satisfies

$$\Delta u+\frac{4}{\kappa^2}e^u(1-e^u)=4\pi\sum_{j=1}^N\delta_{p_j},\quad \text{in } \mathbb{R}^2, \tag{1.3}$$

where $\delta_p(x)$ is the Dirac measure at p. The readers can find the details of the derivation of the above equations in [36, 38] and some recent developments of the related subjects in [3,9,24,31,35,36].

Starting with this paper, we will initiate a comprehensive study of the structure of doubly periodic solutions for (1.3). So we study the following equation

$$\begin{cases}\Delta u+\frac{1}{\varepsilon^2}e^u(1-e^u)=4\pi\sum_{j=1}^N\delta_{p_j}, & \text{in } \Omega\\ u \text{ is doubly periodic on } \partial\Omega,\end{cases} \tag{1.4}$$

where $\varepsilon=\frac{\kappa}{2}>0$ is a small parameter, and Ω is a flat torus in $\mathbb{R}^2$.

Problem (1.4) involves Dirac measures. To eliminate them from the equation, we introduce the Green function $G(x,p)$ of $-\Delta$ in Ω with singularity at p, subject to the doubly periodic boundary condition. That is, $G(x,p)$ satisfies

$$\begin{cases}-\Delta G(x,p)=\delta_p-\frac{1}{|\Omega|}, & \int_\Omega G(x,p)\,dx=0,\\ G(x,p) \text{ is doubly periodic on } \partial\Omega,\end{cases}$$

where $|\Omega|$ is the measure of Ω. Let

$$u_0(x)=-4\pi\sum_{j=1}^N G(x,p_j). \tag{1.5}$$

Using this function u_0, (1.4) is reduced to solving the following problem.

$$\begin{cases}\Delta u+\frac{1}{\varepsilon^2}e^{u+u_0}(1-e^{u+u_0})=\frac{4N\pi}{|\Omega|}, & \text{in } \Omega,\\ u \text{ is doubly periodic on } \partial\Omega.\end{cases} \tag{1.6}$$

Using the maximum principle, we can find that any solution u_ε of (1.6) satisfies $u_\varepsilon+u_0<0$. On the other hand, integrating (1.6) leads to $\int_\Omega e^{u_\varepsilon+u_0}(1-e^{u_\varepsilon+u_0})=\frac{4N\pi\varepsilon^2}{|\Omega|}$, which implies either $u_\varepsilon\to -u_0$, or $u_\varepsilon\to-\infty$ almost everywhere in Ω as $\varepsilon\to 0$. In [10], Choe and Kim proved that (1.6) may have a sequence of solution u_ε, satisfying the following conditions: there is a finite set $\{x_{\varepsilon,1},\cdots,x_{\varepsilon,k}\}$, $x_{\varepsilon,j}\in\Omega$, $j=1,\cdots,k$, such that as $\varepsilon\to0$,

$$u_\varepsilon(x_{\varepsilon,j})+\ln\frac{1}{\varepsilon^2}\to+\infty,\quad \forall\, j=1,\cdots,k, \tag{1.7}$$

and

$$u_\varepsilon(x)+\ln\frac{1}{\varepsilon^2}\to-\infty,\quad \text{uniformly on any compact subset of } \Omega\setminus\{q_1,\cdots,q_k\}, \tag{1.8}$$