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The discrete sign problem: Uniqueness, recovery algorithms and phase retrieval applications

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A B S T R A C T

In this paper we consider the following real-valued and finite dimensional specific instance of the 1-D classical phase retrieval problem. Let $\mathbf{F} \in \mathbb{R}^N$ be an N-dimensional vector, whose discrete Fourier transform has a compact support. The sign problem is to recover $\mathbf{F}$ from its magnitude $|\mathbf{F}|$. First, in contrast to the classical 1-D phase problem which in general has multiple solutions, we prove that with sufficient over-sampling, the sign problem admits a unique solution. Next, we show that the sign problem can be viewed as a special case of a more general piecewise constant phase problem. Relying on this result, we derive a computationally efficient and robust to noise sign recovery algorithm. In the noise-free case and with a sufficiently high sampling rate, our algorithm is guaranteed to recover the true sign pattern. Finally, we present two phase retrieval applications of the sign problem: (i) vectorial phase retrieval with three measurement vectors; and (ii) recovery of two well separated 1-D objects.

1. Introduction

The recovery of a signal from modulus (absolute value) measurements of its Fourier transform, known as *phase retrieval*, is a classical problem with a broad range of applications, including X-ray crystallography [25], astrophysics [18], lensless imaging [24,13], and characterization of ultra-short pulses [36], to name but a few.

From a mathematical perspective, fundamental questions regarding uniqueness of the phase problem and development of reconstruction algorithms have been topics of intense research for several decades. Uniqueness of the phase problem in one and two-dimensions, typically under the assumption that the underlying signal has a compact support, was studied by many authors, see for example [2,9,10,20,30] and

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many additional references therein. For a recent survey and new results on the uniqueness of the 1-D phase problem, see Beinert and Plonka [6]. From the computational aspect, as the classical phase problem is non-convex, the most commonly used phase-retrieval methods are iterative [5,15,17,23]. These often require careful initialization, may exhibit limited robustness to noise, and in general, even in the absence of noise, are not guaranteed to converge to the correct solution.

In recent years there has been a renewed surge of interest in the phase problem. This was motivated in part by proposals of new measurement schemes coupled with novel convex-optimization approaches that provide strong guarantees on correct recovery. Examples include coded diffraction patterns, polarization type schemes and other methods with multiple illuminations [12,3,1,28,26], semi-definite programs for matrix completion [35,11] and sparsity-based recovery methods [31,32].

Motivated by several phase retrieval applications, here we consider a finite dimensional and real-valued particular instance of the general 1-D phase problem, which we denote as the *sign problem*. As described in Section 2, its formulation is as follows: Let $\mathbf{F}$ be an N-dimensional real-valued vector, whose discrete Fourier transform, $\mathbf{f} = \mathcal{DFT}\{\mathbf{F}\} \in \mathbb{C}^N$, has a support of length $\tau + 1$. The sign problem is to recover the sign pattern $\mathbf{s} = sign(\mathbf{F}) \in \{\pm 1\}^N$ from possibly noisy measurements of $|\mathbf{F}|$.

In this paper we perform a detailed study of this finite dimensional sign problem, including its uniqueness and the development of a stable reconstruction algorithm. We also present its application to two practical phase problems. First, in Section 3, Theorem 1 we prove that if $N > 2\tau$, our discrete sign problem admits a unique solution, up to a global ± 1 sign ambiguity. Our proof, similar to [28,9,6], is based on analyzing the roots of high degree polynomials. Since finding such roots is known to be an ill-conditioned problem [34], our proof does not directly lead to a stable reconstruction algorithm.

To robustly solve the sign problem we take a different approach. First, we study the structure of its solutions, showing in Lemma 1 that the sign pattern $\mathbf{s} = sign(\mathbf{F}) \in \{\pm 1\}^N$ cannot be arbitrary, but rather has at most τ sign changes. Next, we relax the constraint that $\mathbf{s} \in \{\pm 1\}^N$ and allow the sign pattern to be a complex-valued N dimensional vector, which guided by Lemma 1, is piecewise constant over at most $\tau + 1$ intervals. This leads us to study the following two questions: (i) is it possible to detect at least parts of these intervals, where the underlying sign pattern is constant? and (ii) does such an *over-segmentation* of $1, \dots, N$ to intervals of constant values indeed retains uniqueness of the problem? With respect to question (ii), we prove in Theorem 2 that given an over-segmentation with few segments M, such that $N > 2\tau + M$, this piecewise constant phase problem has a unique solution.

Based on these theoretical results, in Section 4 we address question (i) above and develop methods to find either an exact or an approximate over-segmentation to intervals of constant sign, given only (noisy) measurements of $|\mathbf{F}|$. Given such an over-segmentation, we then develop a computationally efficient algorithm to retrieve the unknown sign pattern. Our approach follows our previous works [28,29], whereby instead of taking the signal $\mathbf{f}$ as our unknown, we work with the unknown phases, and formulate for them a quadratic functional to be minimized. Relaxing the requirement that the solution is a phase vector leads to solving a system of linear equations. In the noise-free case we prove that with a sufficiently high sampling rate, our algorithm is guaranteed to recover the true sign pattern.

Section 5 presents two phase retrieval applications of practical interest where the sign problem arises. The first is vectorial phase retrieval with three measurement vectors. Here the problem is to recover two compactly supported signals $\mathbf{f}_1$ and $\mathbf{f}_2$ from measurements of $|\mathbf{F}_1|$, $|\mathbf{F}_2|$ and their interference $|\mathbf{F}_1 + \mathbf{F}_2|$. With sufficient over-sampling, this problem was proven to admit a unique solution in [6], but no reconstruction algorithm was given. The second application is the recovery of two well separated 1-D objects from a single spectrum, a problem known to have a unique solution [14]. We show how stable recovery for both problems is possible by solving a related sign problem. Finally, in Section 6 we illustrate the performance of our algorithm via several simulations. For an example with real 2-D experimental data (involving a 2-D sign problem), we refer the reader to [21].

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