ARTICLE IN PRESS
YACHA:1175

Appl. Comput. Harmon. Anal. • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Applied and Computational Harmonic Analysis

www.elsevier.com/locate/acha

Enforcing uniqueness in one-dimensional phase retrieval by additional signal information in time domain

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ARTICLE INFO

Article history:
Received 30 March 2016
Received in revised form 30 September 2016
Accepted 27 December 2016
Available online xxxx
Communicated by Radu Balan

MSC:
42A05
94A08
94A12
94A20

Keywords:
Discrete one-dimensional phase retrieval for complex signals
Compact support
Additional magnitude and phase information in time domain

ABSTRACT

Considering the ambiguousness of the discrete-time phase retrieval problem to recover a signal from its Fourier intensities, one can ask the question: what additional information about the unknown signal do we need to select the correct solution within the large solution set? Based on a characterization of the occurring ambiguities, we investigate different a priori conditions in order to reduce the number of ambiguities or even to receive a unique solution. Particularly, if we have access to additional magnitudes of the unknown signal in the time domain, we can show that almost all signals with finite support can be uniquely recovered. Moreover, we prove that an analogous result can be obtained by exploiting additional phase information.

1. Introduction

The phase retrieval problem consists in recovering a complex-valued signal from the modulus of its Fourier transform. In other words, the phase of the signal in the frequency domain is lost. Recovery problems of this kind have many applications in physics and engineering as for example in crystallography [1–3], astronomy [4], and laser optics [5,6]. Finding an analytic or a numerical solution is generally challenging due to the well-known ambiguousness of the problem. In order to determine a meaningful solution, one hence requires further appropriate information about the unknown signal.

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http://dx.doi.org/10.1016/j.acha.2016.12.002
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There exists a rich literature to characterize and to overcome the occurring ambiguities. Originally, the problem was studied in the continuous-time variant, where one wishes to recover an unknown function $f\colon \mathbb{R} \to \mathbb{C}$ from the given Fourier intensity $|\mathcal{F}[f]|$. Usual additional assumptions are that f has compact support and finite energy. A first characterization of the possibly infinitely many ambiguities has been given in [7,8] and is based on a relation between different solutions described by Blaschke products. Furthermore, the ambiguities can also be characterized by applying Hadamard's factorization theorem for entire functions to the Laplace transform of the autocorrelation function, see for instance [9,10].

In this paper, we consider the phase retrieval problem in a discrete setting and restrict ourselves to the recovery of a complex-valued discrete-time signal $x := (x[n])_{n\in\mathbb{Z}}$ with finite support from its Fourier intensities $|\widehat{x}|$. All occurring ambiguities of this problem can be explicitly constructed via the zeros of the autocorrelation polynomial, see [11–13].

Depending on the application and the exact problem setting, different a priori conditions have been employed in the literature to extract special solutions of the phase retrieval problem. For real-valued signals, the interference with a known reference signal [14,15] can be exploited to reduce the complete solution set to at most two different signals. Further, the interference with an unknown reference signal has been considered in [16]. These approaches can also be used to recover complex-valued signals, see [13,17,18]. Moreover, comparable results have been achieved by special reference signals that are strongly related to the unknown signal itself. For example, one can use a modulated version of the original signal, see [19,20].

Other approaches are based on additional information in the frequency domain. In [21], beside the modulus $|\widehat{x}|$, the information whether the phase of the Fourier transform $\widehat{x}(\omega)$ is contained in $[-\pi/2, \pi/2]$ or in $[-\pi, -\pi/2) \cup (\pi/2, \pi)$ has been studied. One may also replace the Fourier transform by the so-called short-time Fourier transform [22,23], where the original signal is overlapped with a small analysis window at different positions.

If the unknown discrete-time signal has a fixed support of the form $\{0, \dots, M-1\}$ and can thus be identified with an M-dimensional vector, then the Fourier intensity $|\widehat{x}(\omega_k)|$ at different points $\omega_k \in [-\pi, \pi)$ can be written as the intensity measurement $|\langle x, v_k\rangle|$ with $v_k := (\mathrm{e}^{\mathrm{i}\omega_k m})_{m=0}^{M-1}$. In a more general setting using a frame approach, the question arises how the vectors v_k have to be constructed, and how many vectors v_k are needed to ensure a unique recovery of x only from the intensities $|\langle x, v_k\rangle|$, see for instance [20,24–27] and references therein.

A special case of the phase retrieval problem is the recovery of two or more objects with well-separated supports. Here, for two unknown objects, well-separated means that the distance between the objects is greater than the support width of both objects. Although this problem seems more complicated, one can here exploit that the cross-correlations of the different objects are directly encoded in the given Fourier intensity or the given autocorrelation, see [15,28]. A similar result holds for the continuous-time setting [29].

In some applications, like wave front sensing and laser optics [5], besides the Fourier intensity, the modulus of the unknown signal itself is known. For this specific one-dimensional phase retrieval problem, a multi-level Gauss–Newton method has been presented in [6,30,31] to determine a numerical solution. While this method worked well for the considered problems, its stability seems to depend on the given data sets [32]. Further, for some rare cases, the algorithm converges to an approximate solution which is completely different from the original signal.

The occurring numerical problems in the above mentioned multi-level Gauss–Newton method have been the reason for our extensive studies of ambiguities of the complex discrete phase retrieval problem within this paper. Particularly, we will consider the question whether the additional knowledge of $|x| := (|x[n]|)_{n\in\mathbb{Z}}$ can indeed ensure a unique recovery of the unknown signal x. When we understand this uniqueness problem completely, then we can decide whether the behavior of the algorithm is due to ambiguities in the problem setting. Besides modulus constraints in time domain, we will also consider the case that information about the phase $\arg x := (\arg x[n])_{n\in\mathbb{Z}}$ in time domain is available, and investigate how far this additional phase

Please cite this article in press as: R. Beinert, G. Plonka, Enforcing uniqueness in one-dimensional phase retrieval by additional signal information in time domain, Appl. Comput. Harmon. Anal. (2017), http://dx.doi.org/10.1016/j.acha.2016.12.002