ARTICLE IN PRESS YACHA:1183

Appl. Comput. Harmon. Anal. • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Applied and Computational Harmonic Analysis

www.elsevier.com/locate/acha

The distance between the general Poisson summation formula and that for bandlimited functions; applications to quadrature formulae

Paul L. Butzer [a], Gerhard Schmeisser [b], Rudolf L. Stens [a,*]

[a] *Lehrstuhl A für Mathematik, RWTH Aachen, 52056 Aachen, Germany*
[b] *Department of Mathematics, University of Erlangen–Nuremberg, 91058 Erlangen, Germany*

A R T I C L E I N F O

Article history:
Received 4 May 2016
Received in revised form 27 January 2017
Accepted 3 February 2017
Available online xxxx
Communicated by Karlheinz Gröchenig

Dedicated to the memory of Helmut Brass (1936–2011)

MSC:
41A80
65D30
65D32
42A38
46E15
26A16
46E35
26A33

Keywords:
Harmonic analysis
Non-bandlimited functions
Fractional order Riesz derivatives
Lipschitz spaces
Numerical integration
Formulae with remainders
Derivative-free error estimates
Shift-invariant spaces

A B S T R A C T

The general Poisson summation formula of harmonic analysis and analytic number theory can be regarded as a quadrature formula with remainder. The purpose of this investigation is to give estimates for this remainder based on the classical modulus of smoothness and on an appropriate metric for describing the distance of a function from a Bernstein space. Moreover, to be more flexible when measuring the smoothness of a function, we consider Riesz derivatives of fractional order. It will be shown that the use of the above metric in connection with fractional order derivatives leads to estimates for the remainder, which are best possible with respect to the order and the constants.

* Corresponding author.
E-mail addresses: butzer@rwth-aachen.de (P.L. Butzer), schmeisser@mi.uni-erlangen.de (G. Schmeisser), stens@mathA.rwth-aachen.de (R.L. Stens).

http://dx.doi.org/10.1016/j.acha.2017.02.001
1063-5203/

1. Introduction

The general Poisson summation formula, involving a function f and its Fourier transform $\widehat{f}$, as defined in (6), states

$$a \sum_{k\in\mathbb{Z}} f(ak) = \sqrt{2\pi} \sum_{k\in\mathbb{Z}} \widehat{f}(bk) \qquad (a>0,\ b>0,\ ab=2\pi), \tag{1}$$

holding under various assumptions on f.

If f belongs to the Bernstein space B^1_σ (see Section 2 for definitions and notations), then $\widehat{f}(v)$ vanishes outside $(-\sigma,\sigma)$ and (1) takes the particular form ($a = h := 2\pi/\sigma$, $b = \sigma$)

$$\int_{-\infty}^{\infty} f(t)\, dt = h \sum_{k\in\mathbb{Z}} f(hk)\,,$$

which can be interpreted as an exact quadrature formula (trapezoidal rule), as has been observed by many mathematicians since a long time. Moreover, this quadrature formula can be characterized by the features of a Gaussian quadrature formula with respect to Bernstein spaces instead of classical polynomial spaces; for details see [1].

When $f \notin B^1_\sigma$, then this quadrature formula is no longer exact, but one has to add a remainder term. Indeed, the general Poisson formula (1) can be restated as

$$\int_{-\infty}^{\infty} f(t)\, dt = h \sum_{k\in\mathbb{Z}} f(hk) - \sqrt{2\pi} {\sum_{k\in\mathbb{Z}}}' \widehat{f}(k\sigma)\,, \tag{2}$$

the dash at the summation sign indicating that the term for $k = 0$ is omitted.

Poisson's summation formula has many interesting applications. In particular, it is an important tool in the study of shift-invariant space; see, e.g., [2, p. 430], [3, Chap. 13, §7] and [4, p. 65]. Conversely, Poisson's summation formula, or its interpretation as trapezoidal rule on $\mathbb{R}$, can be studied as a special case in the setting of shift-invariant spaces. However, shift-invariant spaces would not simplify our investigations nor would they lead to best possible concrete constants in the error estimates, a major aim of this paper. In fact, one essential improvement in regard to known results in this paper, namely best possible constants in the estimates for the remainders, is the leitmotiv of Korneichuk's well-known book [5], dealing with exact constants for approximation procedures.

A function space $\mathcal{A}$ guaranteeing the validity of (2) and suitable for our purposes is given by

$$\mathcal{A} := \left\{ f \in L^1(\mathbb{R}) \cap C_b(\mathbb{R}); f \in \mathrm{BV}(\mathbb{R}) \text{ or } \widehat{f} \in \mathrm{BV}(\mathbb{R}) \right\},$$

where $\mathrm{BV}(\mathbb{R})$ denotes the set of all functions which are of bounded variation over $\mathbb{R}$. See [6, Section 5.1.5] and [7]. Other conditions for the validity of Poisson's summation formula were given in [8, Theorem 2.25, Corollaries 2.26 and 2.27]. In this paper it is not necessary to look for very weak conditions since our error estimates will need assumptions on f stronger than those specifying $\mathcal{A}$ anyway.

Using the space $\mathcal{A}$, we can state (2) more precisely as follows:

Proposition 1.1. *Let $f \in \mathcal{A}$. Then for $h = 2\pi/\sigma > 0$*

$$\int_{-\infty}^{\infty} f(t)\, dt = h \sum_{k\in\mathbb{Z}} f(hk) + R_\sigma(f), \tag{3}$$

Please cite this article in press as: P.L. Butzer et al., The distance between the general Poisson summation formula and that for bandlimited functions; applications to quadrature formulae, Appl. Comput. Harmon. Anal. (2017), http://dx.doi.org/10.1016/j.acha.2017.02.001