ARTICLE IN PRESS YACHA:1170

Appl. Comput. Harmon. Anal. • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Applied and Computational Harmonic Analysis

www.elsevier.com/locate/acha

A universal formula for generalized cardinal B-splines

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A R T I C L E I N F O

Article history:
Received 9 January 2016
Received in revised form 30 August 2016
Accepted 12 October 2016
Available online xxxx
Communicated by Wolfgang Dahmen

Keywords:
Discrete approximation
Differential operator
Spline
Transfer function

A B S T R A C T

We introduce a universal and systematic way of defining a generalized B-spline based on a linear shift-invariant (LSI) operator L (a.k.a. Fourier multiplier). The generic form of the B-spline is $\beta_{\mathrm{L}} = \mathrm{L_d}\mathrm{L}^{-1}\delta$ where $\mathrm{L}^{-1}\delta$ is the Green's function of L and where $\mathrm{L_d}$ is the discretized version of the operator that has the smallest-possible null space. The cornerstone of our approach is a main construction of $\mathrm{L_d}$ in the form of an infinite product that is motivated by Weierstrass' factorization of entire functions. We show that the resulting Fourier-domain expression is compatible with the construction of all known B-splines. In the special case where L is the derivative operator (linked with piecewise-constant splines), our formula is equivalent to Euler's celebrated decomposition of $\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$ into an infinite product of polynomials. Our main challenge is to prove convergence and to establish continuity results for the proposed infinite-product representation. The ultimate outcome is the demonstration that the generalized B-spline β_{L} generates a Riesz basis of the space of cardinal L-splines, where L is an essentially arbitrary pseudo-differential operator.

1. Introduction

In mathematics and computer graphics, a spline is a function that is piecewise-polynomial of degree n with $n-1$ continuous derivatives [1,2]. The points where the polynomial segments of the spline meet are called knots. The theory of splines was initiated by Schoenberg with the systematic investigation of the cardinal setting where the knots are positioned at the integers [3]. His fundamental result is that a cardinal polynomial spline admits a unique and stable expansion in a B-spline basis where the basis functions are shifted versions of one another. These B-splines are fundamental in the sense that they are the shortest-possible spline constituents. This minimal-support property makes them immensely useful in applications and numerical computations [4].

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http://dx.doi.org/10.1016/j.acha.2016.10.004
1063-5203/

A powerful way of generalizing the notion of spline is by associating a particular brand of spline to some (pseudo) differential operator L [2,5] that, in this work, is assumed to be linear shift-invariant (LSI). Specifically, a function $f(x)$ is called a cardinal L-spline if

$$\mathrm{L}f(x) = \sum_{k\in\mathbb{Z}} a[k]\delta(x-k)$$

where δ is the Dirac impulse and $(a[k])_{k\in\mathbb{Z}}$ is a real-valued sequence of slow growth. The essence of this definition is that the application of the differential operator L uncovers the spline discontinuities: Their location is indicated by the Dirac impulses, which are positioned at the knots, while the size of the discontinuity at $x_k = k$ is encoded in $a[k]$.

The natural question that arises is whether or not Schoenberg's result on the existence of a B-spline basis extends for the generalized brands of L-splines. In other words, we want to investigate the possibility of specifying a generalized B-spline $\beta_{\mathrm{L}}(x)$ that is maximally localized—if possible, of minimum support—and that yields a stable expansion of any cardinal L-spline $f(x)$ such that

$$f(x) = \sum_{k\in\mathbb{Z}} c[k]\beta_{\mathrm{L}}(x-k).$$

So far, the answer has been provided on a case-by-case basis together with rules for constructing B-splines for specific brands of splines. The most prominent examples are:

- Schoenberg's polynomial splines of order m (or degree $(m-1)$) with $\mathrm{L} = \mathrm{D}^m$ where $\mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}x}$ is the derivative operator [3]
- The exponential splines with $\mathrm{L} = \lambda_m \mathrm{D}^m + \cdots + \lambda_1 \mathrm{D} + \mathrm{I}$ (exponential B-splines) [6–9]
- The fractional splines with $\mathrm{L} = \mathrm{D}^\gamma$ where D^γ is the fractional derivative of order $\gamma \in \mathbb{R}^+$ with Fourier symbol $(\mathrm{i}\omega)^\gamma$ [10]

A generic way of expressing β_{L} is

$$\beta_{\mathrm{L}} = \mathrm{L_d}\mathrm{L}^{-1}\delta = \sum_{k\in\mathbb{Z}} d[k]\rho_{\mathrm{L}}(\cdot - k) \tag{1}$$

where $\mathrm{L}^{-1}\delta = \rho_{\mathrm{L}}$ is the Green's function of L and $\mathrm{L_d}$ is a discrete approximation of the operator L (e.g., a finite-difference operator). While (1) gives some insight on the B-spline construction mechanism, it does not resolve the fundamental issue of the determination of an "optimum" set of weights $(d[k])$ for a given operator L; the latter has remained some kind of an art until now, requiring a special treatment for each particular brand of splines. Note that an equivalent form of (1) is

$$\mathrm{L_d}f(x) = \sum_{k\in\mathbb{Z}} d[k]f(x-k) = (\beta_{\mathrm{L}} * \mathrm{L}f)(x)$$

for any continuous and locally integrable function f, which points to the fact that the construction of B-splines is actually equivalent to the design of a "good" discrete approximation $\mathrm{L_d}$ of the operator L. Indeed, since $\mathrm{L_d}f$ is a convolved version of $\mathrm{L}f$ with a B-spline, the best discretization is obtained when the convolution kernel is the most localized and closest (in an appropriate sense) to a Dirac impulse.

The purpose of this paper is to propose a principled approach to this general design problem. The outcome is a general formula for cardinal B-splines that works for a remarkably broad class of (pseudo-)differential operators. These are characterized by a Fourier-domain multiplier $\widehat{L}(\omega)$ that satisfies the admissibility conditions in Definition 2. The proposed framework is consistent with all known constructions and lends

Please cite this article in press as: A. Amini et al., A universal formula for generalized cardinal B-splines, Appl. Comput. Harmon. Anal. (2016), http://dx.doi.org/10.1016/j.acha.2016.10.004