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Letter to the Editor

Similarity matrix framework for data from union of subspaces

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A B S T R A C T

This paper presents a framework for finding similarity matrices for the segmentation of data $\mathbf{W} = [w_1 \cdots w_N] \subset \mathbb{R}^D$ drawn from a union $\mathcal{U} = \bigcup_{i=1}^M S_i$ of independent subspaces $\{S_i\}_{i=1}^M$ of dimensions $\{d_i\}_{i=1}^M$. It is shown that any factorization of $\mathbf{W} = BP$, where columns of B form a basis for data $\mathbf{W}$ and they also come from $\mathcal{U}$, can be used to produce a similarity matrix $\Xi_{\mathbf{W}}$. In other words, $\Xi_{\mathbf{W}}(i,j) \neq 0$, when the columns w_i and w_j of $\mathbf{W}$ come from the same subspace, and $\Xi_{\mathbf{W}}(i,j) = 0$, when the columns w_i and w_j of $\mathbf{W}$ come from different subspaces. Furthermore, $\Xi_{\mathbf{W}} = Q^{d_{max}}$, where $d_{max} = \max\{d_i\}_{i=1}^M$ and $Q \in \mathbb{R}^{N\times N}$ with $Q(i,j) = \left|P^T P(i,j)\right|$. It is shown that a similarity matrix obtained from the reduced row echelon form of $\mathbf{W}$ is a special case of the theory. It is also proven that the Shape Interaction Matrix defined as VV^T, where $\mathbf{W} = U\Sigma V^T$ is the skinny singular value decomposition of $\mathbf{W}$, is not necessarily a similarity matrix. But, taking powers of its absolute value always generates a similarity matrix. An interesting finding of this research is that a similarity matrix can be obtained using a skeleton decomposition of $\mathbf{W}$. First, a square sub-matrix $A \in \mathbb{R}^{r\times r}$ of $\mathbf{W}$ with the same rank r as $\mathbf{W}$ is found. Then, the matrix R corresponding to the rows of $\mathbf{W}$ that contain A is constructed. Finally, a power of the matrix $P^T P$ where $P = A^{-1}R$ provides a similarity matrix $\Xi_{\mathbf{W}}$. Since most of the data matrices are low-rank in many subspace segmentation problems, this is computationally efficient compared to other constructions of similarity matrices.

1. Introduction

In this research, the focus is on the generation of similarity matrices for clustering a set of data points that are drawn from a union of subspaces. Specifically, given a set of data $\mathbf{W} = \{w_1, ..., w_N\} \subset \mathbb{R}^D$ drawn

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from a union of subspaces of the form $\mathcal{U} = \bigcup_{i \in I} S_i$, where $\left\{S_i \subset \mathbb{R}^D\right\}_{i \in I}$ is a set of subspaces, we wish to define a similarity matrix that allows us to

1) determine the number of subspaces $M = |I|$,
2) determine the set of dimension d_i for each subspace S_i,
3) find an orthonormal basis for each subspace S_i,
4) collect the data points belonging to the same subspace into the same cluster.

Union of subspace models have become common in several areas of mathematics and its applications, such as sampling, compressed sensing, and frame theory [1–4]. For example, all images of a given face i with same facial expression, obtained under different illuminations and facial positions, can be modeled as a set of vectors belonging to a low dimensional subspace S_i living in a higher dimensional space $\mathbb{R}^D$ [5,6]. Another example is segmentation of moving rigid objects in videos. Consider a video with F frames of a scene that contains multiple moving rigid objects. Let p be a point on one of these objects and let $x_i(p), y_i(p)$ be the coordinates of p in frame i. Define the *trajectory vector* of p as the vector $w(p) = (x_1(p), y_1(p), x_2(p), y_2(p), \ldots, x_F(p), y_F(p))^T$ in $\mathbb{R}^{2F}$. In this case, the trajectory vectors of multiple independent motions lie in 4-dimensional independent subspaces in $\mathbb{R}^{2F}$ [7,8]. However, it should be noted that independence is a strong assumption for real-world problems (e.g. motion of non-rigid objects or motion of rigid objects on the same planar surface).

In this research, we focus on a way of finding similarity matrices that can be used in clustering algorithms. As such, our aim is to concentrate and understand the first step used in many subspace clustering algorithms. Typically, a method for finding a similarity matrix is used, followed by spectral clustering (a common practice in computer vision and machine learning is to use the absolute value of a similarity matrix as an affinity before spectral clustering). For example, a method related to compressed sensing by Liu et al. [9,10] finds the lowest rank representation of the data matrix. The lowest rank representation is then used to define the similarity of an undirected graph, which is then followed by spectral clustering. It is shown in [11] that the low-rank minimization problem of [9] results in the shape interaction matrix VV^T and the problem is related to factorization rather than sparsity. There are many other methods that produce a similarity matrix as a stage for further processing, such as sparsity methods [12–14,9], algebraic methods [15–17], iterative and statistical methods [18,8,19–22], and spectral clustering methods [13,14,23–29]. Some important methods on subspace clustering are reviewed and their advantages and disadvantages discussed in [28]. Spectral graph partitioning and harmonic analysis on graphs and networks data are explained in [30].

In this research, graph connectivity of data nodes is analyzed to develop theory for a general framework of similarity matrices. Some of the existing techniques generates affinity matrices that has some graph connectivity issues. A discussion on this is given [31]. For example, the well-celebrated Sparse Subspace Clustering (SSC) algorithm produces a sparse affinity matrix whose (i, j)th entry is non-zero only if the data points x_i and x_j are from the same subspaces. However, it is not guaranteed that the data points form the same subspace generates a connected graph. Low-Rank Representation (LRR) method results in an affinity matrix, which is equivalent to Shape Interaction Matrix VV^T [9]. However, in this paper, we show that Shape Interaction Matrix may not always be a similarity matrix and it may lead over-segmentation problem for a set with measure zero.

1.1. Paper contributions

- This paper presents a mathematical framework for finding similarity matrices for segmentating data $\mathbf{W} = [w_1 \cdots w_N] \subset \mathbb{R}^D$ drawn from a union $\mathcal{U} = \bigcup_{i=1}^{M} S_i$ of independent subspaces $\{S_i\}_{i=1}^{M}$ of dimensions $\{d_i\}_{i=1}^{M}$. It is shown that any factorization $\mathbf{W} = BP$, where the columns of B come from $\mathcal{U}$ and form a basis the column space of $\mathbf{W}$, can be used to produce a similarity matrix $\Xi_{\mathbf{W}}$, i.e., if $\Xi_{\mathbf{W}}(i, j) \neq 0$, the columns w_i and w_j of $\mathbf{W}$ come from the same subspace, similarly, and if $\Xi_{\mathbf{W}}(i, j) = 0$, the columns

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