ARTICLE IN PRESS YACHA:1159

Appl. Comput. Harmon. Anal. • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Applied and Computational Harmonic Analysis

www.elsevier.com/locate/acha

On Rayleigh-type formulas for a non-local boundary value problem associated with an integral operator commuting with the Laplacian

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ARTICLE INFO

Article history:
Received 7 March 2016
Accepted 26 August 2016
Available online xxxx
Communicated by Amit Singer

Keywords:
Rayleigh function
Eigenvalues
Non-local boundary value problem
Integral operator
Laplacian

ABSTRACT

In this article we prove the existence, uniqueness, and simplicity of a negative eigenvalue for a class of integral operators whose kernel is of the form $|x - y|^\rho$, $0 < \rho \leq 1$, $x, y \in [-a, a]$. We also provide two different ways of producing recursive formulas for the Rayleigh functions (i.e., recursion formulas for power sums) of the eigenvalues of this integral operator when $\rho = 1$, providing means of approximating this negative eigenvalue. These methods offer recursive procedures for dealing with the eigenvalues of a one-dimensional Laplacian with non-local boundary conditions which commutes with an integral operator having a harmonic kernel. The problem emerged in recent work by one of the authors [48]. We also discuss extensions in higher dimensions and links with distance matrices.

1. Introduction

There has been renewed interest, motivated by applications in statistics, machine learning, and mathematical physics, in the spectral properties of integral operators [5,7–9,12,13,16,25,48]. These operators are usually defined in terms of symmetric distance-like kernels where the focus has recently shifted to questions about spectral embedding, and on establishing connections between empirical operators and their continuous counterparts [47], specifically in the context of manifold learning, with recent activities [6,8,9,13] reviving the theories developed by Schoenberg in the 1930s [49–51], or borrowing techniques from the discrete setting to approximate eigenvalues and eigenfunctions for the continuous counterpart [7,13,46]. As a prototype of such integral operators, we consider

$$\mathscr{K}_{\rho,a} f(x) := C_\rho \int_{-a}^{a} |x - y|^\rho \, f(y) \, \mathrm{d}y \tag{1}$$

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http://dx.doi.org/10.1016/j.acha.2016.08.003
1063-5203/

where $a > 0$ and

$$C_\rho := \frac{\Gamma(-\rho)}{\Gamma\left(\frac{1-\rho}{2}\right)\Gamma\left(\frac{1+\rho}{2}\right)} = \frac{-1}{2\Gamma(1+\rho)\sin\frac{\pi\rho}{2}} \quad < 0 \tag{2}$$

for $0 < \rho \leq 1$, $C_1 = \lim_{\rho\to 1} C_\rho = -1/2$.

The constant C_ρ is motivated by the decomposition of $|x-y|^\rho$, due to Pólya–Szegő [42], who proved that for $-1 \leq x, y \leq 1$, $-1 < \rho < 1$, with $x \neq y$, $\rho \neq 0$,

$$|x-y|^\rho = \frac{\Gamma\left(\frac{1+\rho}{2}\right)\Gamma\left(1-\frac{\rho}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \sum_{n=0}^{\infty} \left(1 - \frac{2n}{\rho}\right) P_n^{(-\frac{\rho}{2})}(x) P_n^{(-\frac{\rho}{2})}(y) \tag{3}$$

(see Eq. (14) of [42], and the comments on p. 29 just before Eq. (18), beginning “Die Entwicklung (14) ...”). They also established the identity

$$\int_{-1}^{1} \left(1-x^2\right)^{-\frac{1+\rho}{2}} |x-y|^\rho P_n^{(-\frac{\rho}{2})}(x)\,\mathrm{d}x = \frac{\Gamma\left(\frac{1-\rho}{2}\right)\Gamma\left(\frac{1+\rho}{2}\right)}{\Gamma(-\rho)} \frac{\Gamma(n-\rho)}{\Gamma(n+1)} P_n^{(-\frac{\rho}{2})}(y). \tag{4}$$

Here $P_n^{(\nu)}(x)$ denotes the ultraspherical (or Gegenbauer) polynomials. In this article we use the classical notation for Gegenbauer polynomials rather than the more modern $C_n^{(\nu)}(x)$ found in e.g., [1, Chap. 22] and [40, Chap. 18]. We also note that the basic properties of the Euler Γ function were used to convert the leading constant in (3) into that in (2). Our choice of C_ρ is tightly connected with (4). For later purposes, we let

$$B_\rho := C_\rho \frac{\Gamma\left(\frac{1+\rho}{2}\right)\Gamma\left(1-\frac{\rho}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} < 0 \tag{5}$$

for $0 < \rho \leq 1$.

In this article we give a direct proof of the existence of a negative eigenvalue for the operator (1), then prove recursion formulas for power sums for its eigenvalues when $\rho = 1$. These power sums provide a means of approximating this unique negative eigenvalue. This problem has arisen in recent work by one of us [48] who developed the theory and applications of an integral operator commuting with the Laplacian defined on a general domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$, satisfying rather interesting non-local boundary condition. In particular, for $d = 1$, as Section 4 reviews this case in detail, the integral operator $\mathscr{K}_{1,1/2}$ defined in (1) was shown to commute with the second order differential operator $-\frac{\mathrm{d}^2}{\mathrm{d}\,x^2}$ with non-local boundary condition. In this article, we focus on the analysis of the spectra of $\mathscr{K}_{\rho,a}$ for $0 < \rho \leq 1$ despite the fact that $\mathscr{K}_{\rho,a}$ with $\rho \neq 1$ does not commute with such a simple 2nd order differential operator and that (3) is also valid for $-1 < \rho < 0$ (see Remark 2.5). The problem is certainly classical, but the results are new. We also show that techniques for the continuous case can be borrowed to provide new proofs for the discrete setting of distance matrices described in [8,9].

We let $L^2[-a, a]$ be the space of square integrable functions on the interval $[-a, a]$. We are interested in the following eigenvalue problem

$$\mathscr{K}_{\rho,a} f(x) = \mu\, f(x). \tag{6}$$