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Hardy space theory on spaces of homogeneous type via orthonormal wavelet bases ☆

Yongsheng Han [a], Ji Li [b], Lesley A. Ward [c,*]

[a] Department of Mathematics, Auburn University, AL 36849-5310, USA
[b] Department of Mathematics, Macquarie University, NSW 2019, Australia
[c] School of Information Technology and Mathematical Sciences, University of South Australia, Mawson Lakes, SA 5095, Australia

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In this paper, we first show that the remarkable orthonormal wavelet expansion for L^p constructed recently by Auscher and Hytönen also converges in certain spaces of test functions and distributions. Hence we establish the theory of product Hardy spaces on spaces $\widetilde{X} = X_1 \times X_2 \times \cdots \times X_n$, where each factor X_i is a space of homogeneous type in the sense of Coifman and Weiss. The main tool we develop is the Littlewood–Paley theory on $\widetilde{X}$, which in turn is a consequence of a corresponding theory on each factor space. We define the square function for this theory in terms of the wavelet coefficients. The Hardy space theory developed in this paper includes product H^p, the dual CMO^p of H^p with the special case $\mathrm{BMO} = \mathrm{CMO}^1$, and the predual VMO of H^1. We also use the wavelet expansion to establish the Calderón–Zygmund decomposition for product H^p, and deduce an interpolation theorem. We make no additional assumptions on the quasi-metric or the doubling measure for each factor space, and thus we extend to the full generality of product spaces of homogeneous type the aspects of both one-parameter and multiparameter theory involving the Littlewood–Paley theory and function spaces. Moreover, our methods would be expected to be a powerful tool for developing wavelet analysis on spaces of homogeneous type.

1. Introduction

We work on wavelet analysis in the setting of product spaces of homogeneous type in the sense of Coifman and Weiss [8], where each factor is of the form (X, d, μ) with d a quasi-metric and μ a doubling measure. We make no additional assumptions on d or μ; such assumptions made in previous related work are discussed

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* Corresponding author.
E-mail addresses: hanyong@auburn.edu (Y. Han), ji.li@mq.edu.au (J. Li), lesley.ward@unisa.edu.au (L.A. Ward).

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below. After recalling the systems of dyadic cubes of Hytönen and Kairema [23] and the orthonormal wavelet basis of Auscher and Hytönen [1], we define an appropriate class of test functions and the induced class of distributions on spaces of homogeneous type. We prove that the Auscher–Hytönen wavelets are test functions, and that the Auscher–Hytönen reproducing formula for L^p also holds for our test functions and distributions. We show that the kernels of certain wavelet operators D_k defined in terms of these wavelets satisfy decay and smoothness conditions similar to those of our test functions. These facts play a crucial role in our development of the Littlewood–Paley theory and function spaces, later in our paper.

We define the discrete Littlewood–Paley square function via the Auscher–Hytönen wavelet coefficients. In order to establish its L^p-boundedness, we also introduce a different, *continuous* Littlewood–Paley square function defined in terms of the wavelet operators D_k. We prove that the discrete and continuous square functions have equivalent norms, by first establishing some inequalities of Plancherel–Pólya type. We develop this Littlewood–Paley theory first in the one-parameter setting, and then for product spaces.

For p in a range that depends on the upper dimensions of the spaces X_1 and X_2 and strictly includes the range $1 \le p < \infty$, we define the product Hardy space $H^p(X_1 \times X_2)$ as the class of distributions whose discrete Littlewood–Paley square functions are in $L^p(X_1 \times X_2)$. (Here we write only two factors, for simplicity, but our results extend to n factors.) For p in this range with $p \le 1$, we define the Carleson measure space $\mathrm{CMO}^p(X_1 \times X_2)$ via the Auscher–Hytönen wavelet coefficients, as a subset of our space of distributions, and prove the duality $(H^p(X_1 \times X_2))' = \mathrm{CMO}^p(X_1 \times X_2)$ by means of sequence spaces that form discrete analogues of these spaces. This duality result includes the special case $(H^1(X_1 \times X_2))' = \mathrm{BMO}(X_1 \times X_2)$. We define the space $\mathrm{VMO}(X_1 \times X_2)$ of functions of vanishing mean oscillation, also in terms of the Auscher–Hytönen wavelet coefficients, and prove the duality $(\mathrm{VMO}(X_1 \times X_2))' = H^1(X_1 \times X_2)$ by adapting an argument of Lacey–Terwilleger–Wick [24]. Using the wavelet expansion, we establish the Calderón–Zygmund decomposition for functions in our Hardy spaces $H^p(X_1 \times X_2)$, again for a suitable range of p that strictly includes $1 \le p < \infty$. As a consequence, we deduce an interpolation theorem for linear operators from these product Hardy spaces to Lebesgue spaces on $X_1 \times X_2$.

We now set our work in context. As Meyer remarked in his preface to [11], *"One is amazed by the dramatic changes that occurred in analysis during the twentieth century. In the 1930s complex methods and Fourier series played a seminal role. After many improvements, mostly achieved by the Calderón–Zygmund school, the action takes place today on spaces of homogeneous type. No group structure is available, the Fourier transform is missing, but a version of harmonic analysis is still present. Indeed the geometry is conducting the analysis."*

Spaces of homogeneous type were introduced by Coifman and Weiss in the early 1970s, in [7]. We say that (X, d, μ) is a *space of homogeneous type* in the sense of Coifman and Weiss if d is a quasi-metric on X and μ is a nonzero measure satisfying the doubling condition. This is the definition used in Coifman and Weiss' paper [8] and in much subsequent work; the original definition given by Coifman and Weiss in [7] was slightly more general. A *quasi-metric* d on a set X is a function $d : X \times X \longrightarrow [0, \infty)$ satisfying (i) $d(x, y) = d(y, x) \ge 0$ for all x, $y \in X$; (ii) $d(x, y) = 0$ if and only if $x = y$; and (iii) the *quasi-triangle inequality*: there is a constant $A_0 \in [1, \infty)$ such that for all x, y, $z \in X$,

$$d(x, y) \le A_0[d(x, z) + d(z, y)]. \tag{1.1}$$

We define the quasi-metric ball by

$$B(x, r) := \{y \in X : d(x, y) < r\} \quad \text{for } x \in X \text{ and } r > 0.$$

Note that the quasi-metric, in contrast to a metric, may not be Hölder regular and quasi-metric balls may not be open. We say that a nonzero measure μ satisfies the *doubling condition* if there is a constant C_μ such that for all $x \in X$ and $r > 0$,