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Letter to the Editor

On sets of large Fourier transform under changes in domain

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A R T I C L E   I N F O

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A B S T R A C T

A function $f : \mathbb{Z}_n \to \mathbb{C}$ can be represented as a linear combination $f(x) = \sum_{\alpha \in \mathbb{Z}_n} \widehat{f}(\alpha)\chi_{\alpha,n}(x)$ where $\widehat{f}$ is the (discrete) Fourier transform of f. Clearly, the basis $\{\chi_{\alpha,n}(x) := \exp(2\pi i \alpha x/n)\}$ depends on the value n.

We show that if f has "large" Fourier coefficients, then the function $\widetilde{f} : \mathbb{Z}_m \to \mathbb{C}$, given by

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{when } 0 \le x < \min(n, m), \\ 0 & \text{otherwise,} \end{cases}$$

also has "large" coefficients. Moreover, they are all contained in a "small" interval around $\lfloor \frac{m}{n}\alpha \rceil$ for each $\alpha \in \mathbb{Z}_n$ such that $\widehat{f}(\alpha)$ is large. One can use this result to recover the large Fourier coefficients of a function f by redefining it on a convenient domain. One can also use this result to reprove a result by Morillo and Ràfols: *single-bit* functions, defined over any domain, are *Fourier concentrated*.

1. Introduction

In recent years different areas of mathematics, such as additive number theory [2], combinatorial number theory [5] and cryptography [4], have seen results that take advantage of the structure arising from large values of the Fourier transform. Most notably, the quantum algorithm for period finding given by Shor [6] is an application that exploits this structure.

In this paper we are concerned with what happens to the Fourier coefficients of a function when we extend or restrict its domain. Our convention is to set the function to be zero on the new added values.

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Definition 1. Let $m, n \in \mathbb{N}$. Let $f : \mathbb{Z}_n \to \mathbb{C}$. Define $\widetilde{f} : \mathbb{Z}_m \to \mathbb{C}$ by

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{when } 0 \le x < \min(n, m), \\ 0 & \text{otherwise.} \end{cases}$$

In this paper we will establish a relationship between the large Fourier coefficients of f and $\widetilde{f}$. Our results are motivated by the following:

1. When $m > n$, our results have some relevance to approximations of functions.
   There is a broad area of research, mostly in the computer science and engineering communities, that studies algorithms to recover the set of points for which the Fourier transform of a given function is large (see [1] for a recent survey). Such algorithms have been extensively studied and optimized for domains of size a power of 2.
   Therefore, our motivation is to show that if the large Fourier coefficients of $\widetilde{f}$ can be recovered for $m = 2^k$, one can then determine the set of large Fourier coefficients of f.
2. When $m < n$, our results apply to functions that can be naturally defined over any domain and the families of functions they give rise to $\{f_n : \mathbb{Z}_n \to \mathbb{C}\}_{n\in\mathbb{N}}$.
   One example of such a family is to take a periodic function $f : \mathbb{Z} \to \mathbb{C}$ and restrict its domain by defining $f_n(x) = f(x)$ for all $x \in \mathbb{Z}_n$. Another example is the family of most-significant-bit functions: for $2^k < n \le 2^{k+1}$, $\mathrm{MSB}_n(x) = 0$ if $x < 2^k$ and $\mathrm{MSB}_n(x) = 1$ otherwise. It turns out that the analysis of the Fourier transform for these functions is much easier on some domains than others.
   Thus, our motivation is to show that by analyzing the large Fourier coefficients of only a small proportion of functions in the family, one can analyze the large Fourier coefficients of all the functions in the family.

1.1. *Definitions and results*

We first give some definitions as well as some basic properties of the Fourier transform.

We use $\lfloor x \rceil$ to denote the closest integer to $x \in \mathbb{R}$ and we use $\|x\|$ for the distance between x and its closest integer, i.e. $\|x\| = |\lfloor x \rceil - x|$. Let $k \in \mathbb{N}$, for all $x \in \mathbb{R}$ define $|x|_k = \min\{|x - kz| \mid z \in \mathbb{Z}\}$, so $|x|_k$ is the distance from x to the nearest integer multiple of k. Equivalently we can define $|x|_k = k\|x/k\|$.

The group $\mathbb{Z}_n$ is the set $\{0, 1, \ldots, n-1\}$ under the group operation of addition modulo n. The set $L^2(\mathbb{Z}_n) = \{f : \mathbb{Z}_n \to \mathbb{C}\}$ of all functions from $\mathbb{Z}_n$ to $\mathbb{C}$ forms a vector space under the usual addition and scalar multiplication of functions. This space has the inner product $\langle f, g\rangle = \frac{1}{n}\sum_{x\in\mathbb{Z}_n} f(x)\overline{g(x)}$, where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$, and norm $\|f\|_2 = \sqrt{\langle f, f\rangle}$.

For each $\alpha \in \mathbb{Z}_n$ the additive character $\chi_{\alpha,n}$ is given by $\chi_{\alpha,n}(x) = \omega_n^{\alpha x}$, where $\omega_n = \exp(2\pi i/n)$ is the n-th root of unity. When it is clear from context what the domain of a character is, we will omit the second index and use χ_α to denote $\chi_{\alpha,n}$. The set of all characters $\{\chi_\alpha\}_{\alpha\in\mathbb{Z}_n}$ forms an orthonormal basis for $L^2(\mathbb{Z}_n)$; thus any function $f : \mathbb{Z}_n \to \mathbb{C}$ can be written as

$$f(x) = \sum_{\alpha\in\mathbb{Z}_n} \widehat{f}(\alpha)\chi_\alpha(x)\,, \tag{1}$$

where $\widehat{f}(\alpha) \in \mathbb{C}$ is the α-th *Fourier coefficient*. Equation (1) implicitly defines a function $\widehat{f} : \mathbb{Z}_n \to \mathbb{C}$, called the Fourier transform of f. It can be explicitly calculated using the formula

$$\widehat{f}(\alpha) = \langle f, \chi_\alpha\rangle = \frac{1}{n}\sum_{x\in\mathbb{Z}_n} f(x)\overline{\chi_\alpha(x)}\,.$$

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