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Practical powerful wavelet packet tests for second-order stationarity

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ABSTRACT

Methods designed for second-order stationary time series can be misleading when applied to nonstationary series, often resulting in inaccurate models and poor forecasts. Hence, testing time series stationarity is important especially with the advent of the 'data revolution' and the recent explosion in the number of nonstationary time series analysis tools. Most existing stationarity tests rely on a single basis. We propose new tests that use nondecimated basis libraries which permit discovery of a wider range of nonstationary behaviours, with greater power whilst preserving acceptable statistical size. Our tests work with a wide range of time series including those whose marginal distributions possess heavy tails. We provide freeware R software that implements our tests and a range of graphical tools to identify the location and duration of nonstationarities. Theoretical and simulated power calculations show the superiority of our wavelet packet approach in a number of important situations and, hence, we suggest that the new tests are useful additions to the analyst's toolbox.

1. Introduction

If a discrete time series, $X_t, t \in \mathbb{Z}$, is stationary then classical (Fourier) theory provides optimal and well-tested means for its analysis and X_t is required to possess the following decomposition:

$$X_t = \int_{-\pi}^{\pi} A(\omega) \exp(i\omega t) d\xi(\omega), \tag{1}$$

where $d\xi(\omega)$ is an orthonormal increments process and $A(\omega)$ is the amplitude function, see, for example, Hannan [33] or Priestley [56].

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We are interested in the case where X_t might be locally stationary: a nonstationary process which appears to be stationarity over short periods but *can* change its statistical properties relatively slowly. Nonstationary time series have been studied over many years, see Page [49] or Silverman [59], for example, and theory was significantly advanced by a series of papers by M.B. Priestley and co-authors from the mid 1960s such as Priestley [55]. Dahlhaus [18] provides a recent review of locally stationary series.

Many nonstationary representations rely on the Fourier basis to provide oscillation: Silverman [59], Priestley [55], Dahlhaus [17], for example. However, for nonstationary processes the Fourier basis and (1) are no longer canonical. For example, Priestley [56] admits general oscillatory basis functions and stipulates conditions on their form and Nason et al. [44] introduced models based on nondecimated wavelets called locally stationary wavelet (LSW) processes and also suggested using wavelet packets to provide oscillation.

If nonstationarities in a time series are not detected, then one will proceed as if the series were stationary — with potentially erroneous results for models and forecasts as stationary analyses average out all the interesting nonstationary behaviour. A simple time series plot can often aid stationarity determination although their interpretation can be somewhat subjective and maybe less desirable than objective rigorous statistical tests.

An early hypothesis test for stationarity was proposed by Priestley and Subba Rao [57] (PSR) that performs an ANOVA analysis on the logarithm of a time-varying spectral estimate at a predefined set of times and frequencies. Software for the PSR test has recently been made publicly available via CRAN in the `fractal` package by Constantine and Percival [16]. Many other tests exist. For example: those that measure correlation between periodogram ordinates such as Hurd and Gerr [35] and Dwivedi and Subba Rao [22], those that measure the discrepancy between a time-varying spectral estimate and its 'closest' stationary spectrum such as Gardner and Zivanovic [31] and Dette et al. [21], and those that measure constancy of some Fourier spectral functional such as Priestley and Subba Rao [57], von Sachs and Neumann [66] and Paparoditis [50]. See also Andrieu and Duvaut [1] for the specific alternative of a Gaussian cyclostationary process. These tests all work with the Fourier spectrum or closely-related quantities.

Tests that work with other bases include Nason [41], which utilised a wavelet spectrum, whereas Cardinali and Nason [8] works with either wavelet or Fourier bases. Another recent alternative is Jin et al. [36] which makes use of the Walsh basis. All these tests have different operational characteristics but it has become evident that nonstationarity can manifest itself in distinctly non-Walsh, non-Fourier and non-wavelet ways. For example, simulations in Nason [41] show that there are cases where (i) neither, (ii) both, or (iii) one or the other of a wavelet or Fourier test successfully detect certain kinds of nonstationarity. Section 2 provides additional evidence as to why measuring in both the Fourier and the wavelet directions alone is unlikely to be enough.

The tests mentioned above measure departures from a *single* type of representation: either Walsh, Fourier or wavelet. A key contribution here is a new approach to testing which looks for departures from *multiple* basis representations. Since our test can look in more directions than one basis it should achieve higher power whilst retaining acceptable size characteristics. In fact, we show this by examining theoretical power in section 5 and via a comprehensive simulation study in section 6.

Section 3 introduces our first stationarity test based on wavelet packets using the 'significant Haar wavelet coefficient' method introduced by von Sachs and Neumann [66] and verifies its theoretical credentials. In practice, this test relies on a hard-to-estimate variance which sometimes results in poor empirical power. Section 4 circumvents this issue by establishing asymptotic equivalence of the Section 3 wavelet packet statistic to the L_2 test statistic from Cardinali and Nason [8] and Dette et al. [21] but assesses significance via a simple bootstrap method.

All our tests are encapsulated in the freeware R software package `BootWPTOS` which also includes graphical tools to identify the location and scale of discovered nonstationarities.

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