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Directional wavelets on n-dimensional spheres

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ABSTRACT

Directional Poisson wavelets, being directional derivatives of Poisson kernel, are introduced on n-dimensional spheres. It is shown that, slightly modified and together with another wavelet family, they are an admissible wavelet pair according to the definition derived from the theory of approximate identities. We investigate some of the properties of directional Poisson wavelets, such as recursive formulae for their Fourier coefficients or explicit representations as functions of spherical variables (for some of the wavelets). We derive also an explicit formula for their Euclidean limits.

1. Introduction

In the present paper we continue our investigation of wavelets over n-dimensional spheres. The definition we use comes from [8] and [17] and it is a generalization of definitions derived from the theory of approximate identities and singular integrals for the two-dimensional case [11,10,9] as well as for the three- and n-dimensional cases [3,5,4,7].

Analysis of anisotropic signals over the two-dimensional sphere is of considerable importance for cosmology, it is useful for the detection and discrimination of non-Gaussianity in cosmic microwave background (CMB) data [23], statistics of temperature anisotropies in the primordial CMB radiation field, or study of maps of the polarization state of the CMB radiation [24]. Another application is presented in [14], where the authors use directional spherical wavelets to the analysis of a world

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map. On the other hand, for texture analysis of crystalline structures which appear in crystals, metals and other materials, non-zonal wavelets over $\mathcal{S}^3$ were introduced in [5]. For such applications, the wavelets defined in the present paper are a well-suited tool. They are a generalization of wavelets over $\mathcal{S}^2$ used by Hayn and Holschneider in [14] to the n-dimensional case, and they suffice the definition given by Ebert et al. in [8]. To the best of our knowledge the present research is the first attempt to define a concrete wavelet family satisfying the conditions of that definition. It is an alternative approach to spherical curvelets and ridgelets presented in [23,24], and its advantage is that no partitioning of the sphere is needed (which could be a problem anyway in more than two dimensions).

The motivation for the choice of Poisson wavelets are their excellent properties in the zonal case (zonal Poisson wavelets are derivatives of Poisson kernel along the axis through the origin of the sphere and the origin of the Poisson kernel [16]): they possess explicit representations and are well-localized [20], they also have discrete frames [21,19] and therefore, are well-suited for computations [15,6]. Also directional wavelets reveal some of that properties, as it is shown in the present paper: the kernel of the wavelet transform is a linear combination of Mexican needlets [22] (and the Gauss–Weierstrass wavelet), thus, it is very well localized; it is straightforward to obtain an explicit representation of the first directional wavelet, cf. Example 4.5, a computation for the second-order wavelet is contained in the Appendix. In a similar way explicit representations for higher-order wavelets can be obtained. Further, the Euclidean limits of directional Poisson wavelets exist and are given by simple formulae, cf. Section 6.

Last but not least, there exist discrete frames of directional wavelets, as it is proven in our forthcoming paper [18], however, this is a property of a much wider class of directional wavelets than Poisson ones.

The paper is organized as follows. Section 2 contains basic information about analysis of functions on spheres. We introduce directional Poisson wavelets as directional derivatives of Poisson kernel in Section 3 and compute recursive formulae for their series representation in Section 4. It is shown in Section 5 that certain linear combinations of that Poisson wavelets satisfy conditions of the definition of wavelets derived from an approximate identity. Finally, Euclidean limit of directional Poisson wavelets are computed in Section 6. It is shown in Appendix how to obtain an explicit representation of a directional Poisson wavelet on the example of the second directional derivative of Poisson kernel.

2. Preliminaries

2.1. Functions on the sphere

By $\mathcal{S}^n$ we denote the n-dimensional unit sphere in $n+1$-dimensional Euclidean space $\mathbb{R}^{n+1}$ with the rotation-invariant measure $d\sigma$ normalized such that

$$\Sigma_n = \int_{\mathcal{S}^n} d\sigma = \frac{2\pi^{(n+1)/2}}{\Gamma\big((n+1)/2\big)}.$$

The surface element $d\sigma$ is explicitly given by

$$d\sigma = \sin^{n-1}\theta_1 \, \sin^{n-2}\theta_2 \ldots \sin\theta_{n-1} d\theta_1 \, d\theta_2 \ldots d\theta_{n-1} d\varphi,$$

where $(\theta_1, \theta_2, \ldots, \theta_{n-1}, \varphi) \in [0, \pi]^{n-1} \times [0, 2\pi)$ are spherical coordinates satisfying

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