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On the reduction of the interferences in the Born–Jordan distribution

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A B S T R A C T

One of the most popular time–frequency representations is certainly the Wigner distribution. To reduce the interferences coming from its quadratic nature, several related distributions have been proposed, among which is the so-called Born–Jordan distribution. It is well known that in the Born–Jordan distribution the ghost frequencies are in fact damped quite well, and the noise is in general reduced. However, the horizontal and vertical directions escape from this general smoothing effect, so that the interferences arranged along these directions are in general kept. Whereas these features are graphically evident on examples and heuristically well understood in the engineering community, there is no at present mathematical explanation of these phenomena, valid for general signals in L^2 and, more in general, in the space $\mathcal{S}'$ of temperate distributions. In the present note we provide such a rigorous study using the notion of wave-front set of a distribution. We use techniques from Time-frequency Analysis, such as the modulation and Wiener amalgam spaces, and also results of microlocal regularity of linear partial differential operators.

1. Introduction

The representation of signals in the time–frequency plane is a fascinating theme involving several mathematical subtleties, mostly related to some form of the uncertainty principle. One of the most popular time–frequency distributions is without any doubt the Wigner distribution, defined by

$$Wf(x,\omega) = \int_{\mathbb{R}^d} f\left(x+\frac{y}{2}\right)\overline{f\left(x-\frac{y}{2}\right)}e^{-2\pi i y\omega}\,dy \qquad x,\omega\in\mathbb{R}^d \tag{1.1}$$

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where the signal f is, say, in the space $\mathcal{S}'(\mathbb{R}^d)$ of temperate distributions in $\mathbb{R}^d$. The quadratic nature of this representation, however, causes the appearance of interferences between several components of the signal. To damp this undesirable effect numerous *Reduced Interference Distributions* have been proposed; see [6,23] for a detailed account. Here we will focus on the Born–Jordan distribution, first introduced in [5], and defined by

$$Qf = Wf * \Theta_\sigma \tag{1.2}$$

where Θ is Cohen's kernel function, given by

$$\Theta(\zeta_1, \zeta_2) = \frac{\sin(\pi\zeta_1\zeta_2)}{\pi\zeta_1\zeta_2}, \quad \zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d} \tag{1.3}$$

($\zeta_1\zeta_2 = \zeta_1 \cdot \zeta_2$ being the scalar product in $\mathbb{R}^d$ and $\Theta(\zeta) = 1$ for $\zeta = 0$), and $\Theta_\sigma(\zeta) = \Theta_\sigma(\zeta_1, \zeta_2) = \mathcal{F}a(\zeta_2, -\zeta_1)$, with $\zeta = (\zeta_1, \zeta_2)$, is the symplectic Fourier transform of Θ, see [2,5–7,9,12,13,23] and the references therein.

To motivate our results and for the benefit of the reader, we now compare (graphically) the features of the Wigner and Born–Jordan distributions of some signals. The following remarks are well known and we refer to [1,2,4,6,22,25,27,28,35,36] and especially to [23] for more details; we also refer to the comprehensive list of references at the end of [23, Chapter 5] for the relevant engineering literature about the geometry of interferences and kernel design.

1.1. Graphical comparisons

Graphical examples both for test signals and real-world signals show that the Born–Jordan distribution, in comparison with the Wigner one, enjoys the following features (in dimension $d = 1$):

a) the so-called "ghost frequencies", arising from the interferences of the several components which do not share the same time or frequency localization, are damped very well.
b) The interferences arranged along the horizontal and vertical direction are substantially kept.
c) The noise is, on the whole, reduced.

These facts can be interpreted (still in dimension $d = 1$) in terms of the following

Principle. *Compared with the Wigner distribution, the Born–Jordan distribution exhibits a general smoothing effect, which however does not involve the horizontal and vertical directions.*

The persistence of interference terms in "vertical" and "horizontal" directions is closely related to the fact that the Born–Jordan distribution preserves the so-called "marginal distributions". As is well-known, the requirement of preserving marginals in Cohen's class implies $\Theta(\zeta_1, \zeta_2)$ to be such that $\Theta(\zeta_1, 0) = 1$ and $\Theta(0, \zeta_2) = 1$, a property which is satisfied by the Born–Jordan kernel. As a result, since two synchronous components with different frequencies exhibit a beating effect ending up with a modulated envelope in time, this must have a signature in the time–frequency plane, and this is exactly given by the "vertical" cross-terms in between the two considered components. By symmetry, the same applies as well "horizontally" to two components that are disjoint in time but in the same frequency band.

The Figs. 1–3 illustrate the above principle. In Fig. 1 the Wigner and Born–Jordan distribution of the sum of 4 Gabor atoms is displayed; notice the presence of 6 interferences (two of which superimposed at the center). In general any two components generate an interference centered in the middle point and arranged along the line joining those components. Notice the disappearance of the "diagonal" interferences in the Born–Jordan distribution.

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