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# A sharp Balian–Low uncertainty principle for shift-invariant spaces

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ABSTRACT

A sharp version of the Balian–Low theorem is proven for the generators of finitely generated shift-invariant spaces. If generators $\{f_k\}_{k=1}^K \subset L^2(\mathbb{R}^d)$ are translated along a lattice to form a frame or Riesz basis for a shift-invariant space $V$, and if $V$ has extra invariance by a suitable finer lattice, then one of the generators $f_k$ must satisfy $\int_{\mathbb{R}^d} |x|\,|f_k(x)|^2 dx = \infty$, namely, $\widehat{f_k} \notin H^{1/2}(\mathbb{R}^d)$. Similar results are proven for frames of translates that are not Riesz bases without the assumption of extra lattice invariance. The best previously existing results in the literature give a notably weaker conclusion using the Sobolev space $H^{d/2+\epsilon}(\mathbb{R}^d)$; our results provide an absolutely sharp improvement with $H^{1/2}(\mathbb{R}^d)$. Our results are sharp in the sense that $H^{1/2}(\mathbb{R}^d)$ cannot be replaced by $H^s(\mathbb{R}^d)$ for any $s < 1/2$.



## 1. Introduction

The uncertainty principle in harmonic analysis is a class of results which constrains how well-localized a function $f$ and its Fourier transform $\widehat{f}$ can be. A classical expression of the uncertainty principle is given by the $d$-dimensional Heisenberg inequality

$$\forall f \in L^2(\mathbb{R}^d), \qquad \left(\int_{\mathbb{R}^d} |x|^2 |f(x)|^2 dx\right)\left(\int_{\mathbb{R}^d} |\xi|^2 |\widehat{f}(\xi)|^2 d\xi\right) \geq \frac{d^2}{16\pi^2}\|f\|^4_{L^2(\mathbb{R}^d)}, \tag{1.1}$$

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where the Fourier transform $\widehat{f} \in L^2(\mathbb{R}^d)$ is defined using $\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot\xi}dx$. For background on this and other uncertainty principles, see [26,32].

There exist versions of the uncertainty principle which not only constrain time and frequency localization of an individual function as in (1.1), but instead constrain the collective time and frequency localization of orthonormal bases and other structured spanning systems such as frames and Riesz bases. A collection $\{h_n\}_{n=1}^\infty$ in a Hilbert space $\mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exist constants $0 < A \le B < \infty$ such that

$$\forall h \in \mathcal{H}, \quad A\|h\|_{\mathcal{H}}^2 \le \sum_{n=1}^{\infty} |\langle h, h_n\rangle_{\mathcal{H}}|^2 \le B\|h\|_{\mathcal{H}}^2.$$

The collection $\{h_n\}_{n=1}^\infty$ is a *Riesz basis* for $\mathcal{H}$ if it is a minimal frame for $\mathcal{H}$, i.e., $\{h_n\}_{n=1}^\infty$ is a frame for $\mathcal{H}$ but $\{h_n\}_{n=1}^\infty\backslash\{h_N\}$ is not a frame for $\mathcal{H}$ for any $N \ge 1$. Equivalently, $\{h_n\}_{n=1}^\infty$ is a Riesz basis for $\mathcal{H}$ if and only if $\{h_n\}_{n=1}^\infty$ is the image of an orthonormal basis under a bounded invertible operator from $\mathcal{H}$ to $\mathcal{H}$. Every orthonormal basis is automatically a Riesz basis and a frame, but there exist frames that are not Riesz bases, and Riesz bases that are not orthonormal bases. See [22] for background on frames and Riesz bases.

The following beautiful example of an uncertainty principle for Riesz bases was proven in [30]. If $\{f_n\}_{n=1}^\infty \subset L^2(\mathbb{R}^d)$ satisfies

$$\sup_n \left( \int_{\mathbb{R}^d} |x - a_n|^{2d+\epsilon} |f_n(x)|^2 dx \right) \left( \int_{\mathbb{R}^d} |\xi - b_n|^{2d+\epsilon} |\widehat{f_n}(\xi)|^2 d\xi \right) < \infty, \tag{1.2}$$

for some $\epsilon > 0$ and $\{(a_n, b_n)\}_{n=1}^\infty \subset \mathbb{R}^2$, then $\{f_n\}_{n=1}^\infty$ cannot be a Riesz basis for $L^2(\mathbb{R}^d)$. Moreover, this result is sharp in that $\epsilon$ cannot be taken to be zero, see [17,30].

There has been particular interest in uncertainty principles for bases that are endowed with an underlying group structure. The Balian–Low theorem for Gabor systems is a celebrated result of this type. Given $f \in L^2(\mathbb{R})$ the associated *Gabor system* $\mathcal{G}(f, 1, 1) = \{f_{m,n}\}_{m,n\in\mathbb{Z}}$ is defined by $f_{m,n}(x) = e^{2\pi i m x} f(x - n)$. The following nonsymmetric version of the Balian–Low theorem states that if $\mathcal{G}(f, 1, 1)$ is a Riesz basis for $L^2(\mathbb{R})$ then $f$ must be poorly localized in either time or frequency.

**Theorem 1.1** *(Balian–Low theorems). Let $f \in L^2(\mathbb{R})$ and suppose that $\mathcal{G}(f, 1, 1)$ is a Riesz basis for $L^2(\mathbb{R})$.*

*(1) If $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\left( \int_{\mathbb{R}} |x|^p |f(x)|^2 dx \right) \left( \int_{\mathbb{R}} |\xi|^q |\widehat{f}(\xi)|^2 d\xi \right) = \infty.$$

*(2) If $\widehat{f}$ is compactly supported, then*

$$\int_{\mathbb{R}} |x|\, |f(x)|^2 dx = \infty.$$

*The same result holds with the roles of $f$ and $\widehat{f}$ interchanged.*

The original Balian–Low theorem [6,41] formulated the case $p = q = 2$ in part (1) of Theorem 1.1 for orthonormal bases. The non-symmetrically weighted $(p, q)$ versions with $p \ne q$ in Theorem 1.1 were subsequently proven in [28]. There are numerous extensions of the Balian–Low theorem, e.g., see the surveys [13,23] and articles [4,5,8–12,24,27,31,34,35,39,40,42,43].

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