Accepted Manuscript

On the numerical solution of second order ordinary differential equations in the high-frequency regime

James Bremer

PII: S1063-5203(16)30018-5
DOI: http://dx.doi.org/10.1016/j.acha.2016.05.002
Reference: YACHA 1139

To appear in: *Applied and Computational Harmonic Analysis*

Received date: 27 November 2015
Revised date: 29 March 2016
Accepted date: 15 May 2016

Please cite this article in press as: J. Bremer, On the numerical solution of second order ordinary differential equations in the high-frequency regime, *Appl. Comput. Harmon. Anal.* (2016), http://dx.doi.org/10.1016/j.acha.2016.05.002

On the numerical solution of second order ordinary differential equations in the high-frequency regime

James Bremer[a,*]

[a] *Department of Mathematics, University of California, Davis*

Abstract

We describe an algorithm for the numerical solution of second order linear ordinary differential equations in the high-frequency regime. It is based on the recent observation that solutions of equations of this type can be accurately represented using nonoscillatory phase functions. Unlike standard solvers for ordinary differential equations, the running time of our algorithm is independent of the frequency of oscillation of the solutions. We illustrate this and other properties of the method with several numerical experiments.

Keywords: Ordinary differential equations, fast algorithms, phase functions, special functions, Bessel's equation

1. Introduction

Second order linear differential equations of the form

$$y''(t) + \lambda^2 q(t) y(t) = 0 \quad \text{for all} \quad a \leqslant t \leqslant b \tag{1}$$

are ubiquitous in analysis and mathematical physics. As a consequence, much attention has been devoted to the development of numerical algorithms for their solution and, in most regimes, fast and accurate methods are available.

However, when q is positive and λ is real-valued and large, the solutions of (1) are highly oscillatory (this is a consequence of the Sturm comparison theorem) and standard solvers for ordinary differential equations (for instance, Runge-Kutta schemes and spectral methods) suffer. Specifically, their running times grow linearly with the parameter λ, which makes them prohibitively expensive when λ is large.

Because of the poor performance of standard solvers, asymptotic methods are often used in this regime. In some instances, they allow for the accurate evaluation of solutions of equation of the form (1) using a number of operations which is independent of the parameter λ. For example, [3] presents an $\mathcal{O}(1)$ algorithm for calculating Legendre polynomials of arbitrary order using a combination of direct evaluation and asymptotic formulas; it achieves near machine precision accuracy and serves as the basis for a fast algorithm (also presented in [3]) for the construction of Gauss-Legendre quadratures of extremely large orders. In a similar vein, [14] describes a fast algorithm for the computation of Gauss-Legendre and Gauss-Jacobi quadratures which makes use of asymptotic formulas in order to evaluate Jacobi polynomials in $\mathcal{O}(1)$ operations.

*Corresponding author (bremer@math.ucdavis.edu)