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Letter to the Editor

The minimal measurement number for low-rank matrix recovery

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A R T I C L E I N F O

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A B S T R A C T

The paper presents several results that address a fundamental question in low-rank matrix recovery: how many measurements are needed to recover low-rank matrices? We begin by investigating the complex matrices case and show that $4nr-4r^2$ generic measurements are both necessary and sufficient for the recovery of rank-r matrices in $\mathbb{C}^{n\times n}$. Thus, we confirm a conjecture which is raised by Eldar, Needell and Plan for the complex case. We next consider the real case and prove that the bound $4nr-4r^2$ is tight provided $n=2^k+r, k\in\mathbb{Z}_+$. Motivated by Vinzant's work [19], we construct 11 matrices in $\mathbb{R}^{4\times 4}$ by computer random search and prove they define injective measurements on rank-1 matrices in $\mathbb{R}^{4\times 4}$. This disproves the conjecture raised by Eldar, Needell and Plan for the real case. Finally, we use the results in this paper to investigate the phase retrieval by projection and show fewer than $2n-1$ orthogonal projections are possible for the recovery of $x\in\mathbb{R}^n$ from the norm of them, which gives a negative answer for a question raised in [1].

1. Introduction

1.1. Problem setup

The problem of low-rank matrix recovery attracted many attention recently since it is widely used in image processing, system identification and control, Euclidean embedding, and recommender systems. Suppose that the matrix $Q\in\mathbb{H}^{n\times n}$ with $\mathrm{rank}(Q)\le r$, where $\mathbb{H}$ is either $\mathbb{R}$ or $\mathbb{C}$. The information we gather about Q is

$$b_j := \langle A_j, Q\rangle := \mathrm{trace}(A_j^* Q), \quad j=1,\dots,m$$

where $A_j\in\mathbb{H}^{n\times n}, j=1,\dots,m$. The aim of the low-rank matrix recovery is to recover Q from $\mathbf{b}=[b_1,\dots,b_m]\in\mathbb{H}^m$.

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For a given $\mathcal{A} := \{A_1, \ldots, A_m\} \subset \mathbb{H}^{n\times n}$, we define the map $\mathbf{M}_\mathcal{A} : \mathbb{H}^{n\times n} \to \mathbb{H}^m$ by

$$\mathbf{M}_\mathcal{A}(Q) = [b_1, \ldots, b_m].$$

Set

$$\mathcal{L}_r^\mathbb{H} := \{X \in \mathbb{H}^{n\times n} : \mathrm{rank}(X) \le r\}.$$

We say the matrices set $\mathcal{A} := \{A_1, \ldots, A_m\}$ has the low-rank matrix recovery property for $\mathcal{L}_r^\mathbb{H}$ if the map $\mathbf{M}_\mathcal{A}$ is injective on $\mathcal{L}_r^\mathbb{H}$. Naturally, we are interested in the minimal m for which the map $\mathbf{M}_\mathcal{A}$ is injective on $\mathcal{L}_r^\mathbb{H}$.

There are many convex programs for the recovery of low-rank matrices Q from $\mathbf{M}_\mathcal{A}(Q)$. A well-known one is nuclear-norm minimization which requires $m = Cnr$ random linear measurements for the recovery of rank-r matrices in $\mathbb{H}^{n\times n}$ [6–8,16]. Despite many literatures on low-rank matrix recovery, there remains a fundamental lack of understanding about the theoretical limit of the number of the cardinality of $\mathcal{A}$ which has the low-rank matrix recovery property for $\mathcal{L}_r^\mathbb{H}$. This paper focusses on the problem of the minimal measurements number for the recovery of low-rank matrices. We state the problem as follows:

Problem 1 What is the minimal measurement number m for which there exists $\mathcal{A} = \{A_1, \ldots, A_m\} \subset \mathbb{H}^{n\times n}$ so that $\mathbf{M}_\mathcal{A}$ is injective on $\mathcal{L}_r^\mathbb{H}$?

The aim of this paper is to addresses Problem 1 under many different settings.

1.2. Related work

A related problem to low-rank matrix recovery is *phase retrieval*, which is to recover the rank-one matrix $xx^* \in \mathbb{H}^{n\times n}$ from the measurements $|\langle \phi_j, x\rangle|^2 = \langle \phi_j\phi_j^*, xx^*\rangle, j = 1, \ldots, m$, where $\phi_j \in \mathbb{H}^n$ and $x \in \mathbb{H}^n$. In the context of phase retrieval, one is interested in the minimal measurement number m for which the map $\mathbf{M}_\Phi$ is injective on $\mathcal{S}_1^\mathbb{H}$ where $\Phi := \{\phi_1\phi_1^*, \ldots, \phi_m\phi_m^*\}$ and

$$\mathcal{S}_r^\mathbb{H} := \{X \in \mathbb{H}^{n\times n} : \mathrm{rank}(X) \le r, X^* = X\}, \quad r \in \mathbb{Z}.$$

It is known that in the real case $\mathbb{H} = \mathbb{R}$ one needs at least $m \ge 2n - 1$ vectors so that $\mathbf{M}_\Phi$ is injective on $S_1^\mathbb{R}$ [2]. For the complex case $\mathbb{H} = \mathbb{C}$, the same problem remain open. Balan, Casazza and Edidin in [2] show that $\mathbf{M}_\Phi$ is injective on $S_1^\mathbb{C}$ if $m \ge 4n - 2$ and $\phi_1, \ldots, \phi_m$ are generic vectors in $\mathbb{C}^n$. In [3], Bandeira, Cahill, Mixon and Nelson conjectured the following (a) if $m < 4n-4$ then $\mathbf{M}_\Phi$ is not injective on $\mathcal{S}_1^\mathbb{C}$; (b) if $m \ge 4n - 4$ then $\mathbf{M}_\Phi$ is injective on $\mathcal{S}_1^\mathbb{C}$ for generic vectors $\phi_j, j = 1, \ldots, m$. The part (b) of the conjecture is proved by Conca, Edidin, Hering and Vinzant in [9] by employing algebraic tools. They also confirm part (a) for the case where n is in the form of $2^k + 1, k \in \mathbb{Z}$. Recently, in [19], a counterexample is presented disproving part (a) of this conjecture. In fact, [19] gives $11 = 4n - 5 < 4n - 4$ vectors $\phi_1, \ldots, \phi_{11} \in \mathbb{C}^4$ and prove that $\mathbf{M}_\Phi$ is injective on $\mathcal{S}_1^\mathbb{C}$ by algebraic computation where $\Phi = \{\phi_1\phi_1^*, \ldots, \phi_{11}\phi_{11}^*\}$.

In the context of low-rank matrix recovery, it is Eldar, Needell and Plan [11] that show that $m \ge 4nr-4r^2$ Gaussian matrices $A_1, \ldots, A_m$ have low-rank matrix recovery property for $\mathcal{L}_r^\mathbb{H}$ with probability 1 (see also [13,14,18]) provided $r \le n/2$. Naturally, one may be interested in whether the number $4nr - 4r^2$ is tight. In [11], the authors made the following conjecture:

Conjecture 1.1. *[11] If $m < 4nr - 4r^2$ then $\mathbf{M}_\mathcal{A}$ is not injective on $\mathcal{L}_r^\mathbb{H}$.*

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