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A class of fully nonlinear equations on the closed manifold ☆

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A generalized k-Yamabe problem is considered in this paper. Denoting Ric and R the Ricci tensor and the scalar curvature of a Riemannian space (M, g) respectively, we consider the σ_k-type equation $\sigma_k(\lambda_{st}) = const.$, where λ_{st} are the eigenvalues of the symmetric tensor $sRic - tR \cdot g$ and σ_k is the $k - th$ elementary symmetric polynomial. We show that the equation is solvable in a conformal class if $sRic - tR \cdot g$ is in the convex cone Γ_k^+ and $2t > s > 0$.

1. Introduction

Let (M^n, g), for $n \geq 3$, be a closed Riemannian manifold. Denote the Ricci tensor and the scalar tensor by Ric and R (or R_g), respectively. Consider the following combination type tensor

$$T_t^s := sRic - tR_g \cdot g$$

on M, where s, t are some constants. Clearly, T_0^1 is just the Ricci tensor, $T_{1/2(n-1)(n-2)}^{1/(n-2)}$ is the Schouten tensor and $T_{1/2(n-2)}^{1/(n-2)}$ is the Einstein tensor.

Let $[g]$ be the set of the metrics conforming to g, and $\lambda(T_t^s)$ be the eigenvalue of $g^{-1}T_t^s$. The prescribed k-curvature problems of T_t^s is to find a metric $\tilde{g} \in [g]$ satisfying the following equation

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$$\sigma_k\left(\lambda\left(\tilde{T}_t^s\right)\right) = const., \tag{1.1}$$

where $1 \le k \le n$ is an integer, $\tilde{T}_t^s$ is the tensor with respect to $\tilde{g}$, and for all $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$, the elementary symmetric function σ_k: $\mathbb{R}^n \to \mathbb{R}$ is defined by

$$\sigma_k(\lambda) = \sigma_k(\lambda_1, \cdots, \lambda_n) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}.$$

When $k = 1$ and $s \neq nt$, Equation (1.1) is just the well known Yamabe problem (up to a constant), which has been solved by Yamabe, Trudinger, Aubin and Schoen (see [9]). For $k \ge 2$, $s = \frac{1}{n-2}$ and $t = \frac{s}{2(n-1)}$, Equation (1.1) is the k-Yamabe problem which was introduced by Gursky–Viaclovsky in 2003 [5] and has been extensively studied (see [1,2,4,6,11,12,14] etc.).

Let $\tilde{g} = e^{2u}g$, where u is a function defined on M. Then T_t^s transforms according to the formula

$$\begin{aligned} \tilde{T}_t^s = {} & [2t(n-1) - s]\triangle u g - s(n-2)\nabla^2 u + s(n-2)\, du \otimes du \\ & + [t(n-1) - s](n-2)|\nabla u|^2 \cdot g + T_t^s, \end{aligned} \tag{1.2}$$

where ∇u, $\nabla^2 u$ denote the gradient and Hessian of u with respect to g, respectively. Hence, the problem (1.1) is corresponding to the following equation

$$\sigma_k^{\frac{1}{k}}\left(\lambda\left[\begin{matrix}[2t(n-1) - s]\triangle u g - s(n-2)\nabla^2 u + s(n-2)\, du \otimes du \\ +[t(n-1) - s](n-2)|\nabla u|^2 g + T\end{matrix}\right]\right) = e^{2u} \cdot const. \tag{1.3}$$

Clearly, (1.3) is a fully nonlinear partial differential equation when $k \ge 2$.

To solve Equation (1.3), we will consider a more general equation in this paper.

Let

$$\Gamma_k^+ = \{\lambda \in \mathbb{R}^n | \sigma_j(\lambda) > 0, \forall 1 \le j \le k\}.$$

Clearly, $\Gamma_n^+ \subset \Gamma_{n-1}^+ \subset \cdots \subset \Gamma_1^+$. Let Γ^+ be an open convex cone in $\mathbb{R}^n$ with vertex at the origin and satisfy $\Gamma_n^+ \subset \Gamma^+ \subset \Gamma_1^+$.

Let $F : \mathbb{R}^n \to \mathbb{R}$ be a smooth symmetric function, homogeneous of degree one and satisfy the following in Γ^+:

(A1) F is positive, and $F = 0$ on $\partial\Gamma^+$;

(A2) F is monotone (i.e., $\frac{\partial F}{\partial \lambda_i}$ is positive);

(A3) F is concave (i.e., $\frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j}$ is negative semi-definite).

By the argument in [13], we also have

$$\sum_i \frac{\partial F}{\partial \lambda_i}(\lambda) \ge F(e) > 0 \text{ in } \Gamma^+, \tag{A4}$$

where $e = (1, 1, \cdots, 1) \in \mathbb{R}^n$. It is well known that $\sigma_k^{\frac{1}{k}}$ satisfies all the conditions above on Γ_k^+ (see [10]).

Given a smooth function $\varphi(x, z) : M^n \times \mathbb{R} \to \mathbb{R}$, we consider the following equation

$$F\left(\lambda\left(\alpha\triangle u g - \beta\nabla^2 u + a(x)\, du \otimes du + b(x)|\nabla u|^2 g + B\right)\right) = \varphi(x, u), \tag{1.4}$$

where α, β are constants, $a, b \in C^\infty(M)$, and B is a smooth symmetric $(0, 2)$-tensor on M. For convenience, we define